S.-T. Yau College Student Mathematics Contests 2023

Mathematical Physics

Solve every problem.

1. Consider the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\frac{\dot{x}^2 + \dot{y}^2 + 2(x\dot{y} - y\dot{x})}{x^2 + y^2} \tag{1}$$

   (a) Compute the Hamiltonian $H(x, y, p_z, p_y)$, and show the final form can be written as

$$\frac{1}{2}f(x, y)[(p_x - A_x(x, y))^2 + (p_y - A_y(x, y))^2] \tag{2}$$

   for some f, A_x, A_y. Find the vector potential $\vec{A}$ and then compute the corresponding magnetic field away from the origin. (Hint: recall that $\vec{B} = \mathrm{curl}\vec{A} = \nabla \times \vec{A}$).

   (b) Prove that the Lagrangian $L(x, y, \dot{x}, \dot{y})$ is invariant under two symmetries: rotations and scale transformations.

   (c) Derive the conserved quantities for both symmetries.

   (d) Rewrite the Lagrangian in polar coordinates, write down the Euler-Lagrange equations and solve them.

2. A particle of mass m in 2 dimensions is confined by an isotropic harmonic oscillator potential of frequency ω, while subject to a weak and anisotropic perturbation of strength $\alpha \ll 1$. The total Hamiltonian of the particle is

$$H = H_0 + V = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) + \alpha m\omega^2 xy \tag{3}$$

   (a) When $\alpha = 0$, what are the energies and degeneracies of the three lowest-lying unperturbed states?

   (b) Use perturbation theory to correct the energies of the above three states to the first order in α.

   (c) Find the exact spectrum of H. (Hint: you may want to rotate x and y into a new coordinates)

   (d) Check that the perturbative results in part b. are recovered from the exact spectrum.

3. One can express the electric fields $\vec{E}$ and magnetic fields $\vec{B}$ in terms of the scalar and vector potentials, $A^\mu = (\phi, \vec{A})$.

   (a) Write down the expression of $\vec{E}$ and $\vec{B}$ in terms of $(\phi, \vec{A})$ and show that the result is unchanged under gauge transformation

   $$\phi \to \phi + \frac{\partial}{\partial t} f, \quad \vec{A} \to \vec{A} - \nabla f, \tag{4}$$

   where $f = f(\vec{x}, t)$ is a scalar function.

   (b) Show that two of the 4 Maxwell equations of $\vec{E}$ and $\vec{B}$ are satisfied automatically in terms of $A^\mu = (\phi, \vec{A})$.

   (c) Derive the equations for the scalar and vector potentials from the remaining Maxwell equations in Lorentz gauge.

   $$\frac{1}{c}\partial_t \phi + \nabla \cdot \vec{A} = 0, \tag{5}$$

   (d) Recall that the Green's function of the wave equation

   $$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) G(t, \vec{r}) = \delta(t)\delta^3(\vec{r}) \tag{6}$$

   is

   $$G(t - t_0, \vec{r} - \vec{r}_0) = \frac{\theta(t - t_0)}{4\pi|\vec{r} - \vec{r}_0|}\delta\left(t - t_0 - \frac{|\vec{r} - \vec{r}_0|}{c}\right). \tag{7}$$

   Assume a particle of electric charge e moves with trajectory $\vec{R}(t)$ with $\vec{v}(t) = d\vec{R}(t)/dt$. Use the Green's function to derive the potential $\phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ of this particle at $(\vec{r}, t)$. You may assume that $|\vec{R}(t)| \ll |\vec{r}|$, $|\vec{R}(t)| \ll ct$ and $|\vec{v}(t)| \ll c$ and expand your result up to the order $\mathcal{O}(1/|\vec{r}|)$ and $\mathcal{O}(|\vec{v}(t)|/c)$. This is also called non-relativistic and far-field approximations.

4. Consider a one-dimensional system of free massless bosons with one polarization, and the dispersion relation $E_k = \hbar v|k|$, where v is the particle velocity, k wavevector, E_k energy. The particles are not interacting either among themselves or with external scattering potentials. If the system is in equilibrium at temperature T,

   (a) Assume that the chemical potential μ is 0, calculate the heat capacity C per unit length.

   (b) Repeat the calculations for massive fermions, for which $E_k = \hbar^2 k^2/2m$ and the chemical potential μ is far above the bottom of the energy spectrum: $\mu \gg k_B T$. (Consider one spin direction for the fermions, and you may make reasonable approximations.)

Hint: you might find the following formulae useful to evaluate some integrals needed

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}. \tag{8}$$

5. Consider the vacuum Einstein's equation in four dimensional spacetime with a cosmological constant

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0. \tag{9}$$

   (a) Proof that $R_{\mu\nu} = k g_{\mu\nu}$ and find out the value of k.

   (b) Now start with an ansatz of a metric in the following form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{10}$$

   where $f(r)$ is a **polynomial** in r. Compute non zero components of the Ricci tensor $R_{\mu\nu}$ and scalar curvature R of this metric.

   (c) Assuming that the above ansatz is a solution of the vacuum Einstein equation with cosmological constant Λ, solve $f(r)$.

   (d) Prove that ∂_t and ∂_ϕ are Killing vector fields.

6. Consider the ϕ^3 model with a real scalar field $\phi(x)$ in $3+1$ dimensional Minkowski spacetime with metric $(-,+,+,+)$. Its Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{6}g\phi^3, \tag{11}$$

   where g is a coupling with dimensions of mass.

   (a) Write down the propagator and the interaction vertex for this model in momentum space.

   (b) Compute the one-loop self-energy graph using dimensional regularization.

   (c) Introducing $m^2 = m_R^2 + \delta m^2$. What is the value of δm^2 if we want to write the one-loop self-energy graph as a finite function of m_R?

   You may find the following formula useful

$$(AB)^{-1} = \int_0^1 dx[xA + (1-x)B]^{-2}, \tag{12}$$

$$\int d^dk \frac{1}{(-k^2 - 2p\cdot k - M^2 + i\epsilon)^s} = (-1)^s i\pi^{d/2} \frac{\Gamma(s-d/2)}{\Gamma(s)} (-p^2 + M^2 - i\epsilon)^{d/2-s}, \tag{13}$$

where $\Gamma(z)$ is the Gamma function which has a simple pole at the origin.

$$\Gamma(z) = \frac{1}{z} - \gamma + \mathcal{O}(z), \tag{14}$$

where γ is the Euler constant.