S.-T. Yau College Student Mathematics Contests 2019

Applied and Computational Math

Team (5 problems)

1) (10 points)

Show that the quadrature formula $\displaystyle\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{n} \sum_{k=0}^{n-1} f\left(\cos\pi\frac{2k+1}{2n}\right)$ is exact for all polynomials of degree up to and including $2n-1$.

2) (15 pointes) Let $x = (x_0, \ldots, x_{N-1}) \in \mathbf{R}^N$, $x \neq 0$ and $\hat{x}$ be its discrete Fourier transform, i.e.

$$\hat{x}_w = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x_t \exp(-2\pi i w t/N),\ w = 0, \ldots, N-1.$$

Prove that $\|x\|_0 \|\hat{x}\|_0 \geqslant N$ where $\|x\|_0$ denotes the number of nonzero entries in x. (Hint: show that $\hat{x}$ can not have $\|x\|_0$ consecutive zeros.)

3) (20 pointes)

Let $m \leqslant n$. Consider the $(n+m) \times (n+m)$ real matrix defined by

$$A = \begin{bmatrix} I & X \\ X^\top & O \end{bmatrix},$$

where I is the $n \times n$ identity matrix, X is a full-rank $n \times m$ matrix, O is the $m \times m$ zero matrix.

(i) Show that A is nonsingular.

(ii) Find the eigenvalues of A, some of which are in terms of the singular values of X.

(iii) Under what conditions on X would the iteration

$$x_{n+1} = x_n - (Ax_n - b)$$

converge to the solution of $Ax = b$ for any $(n+m) \times (n+m)$ real vector b?

4) (25 pointes)

Let f be a continuously differentiable convex function defined on $\mathbf{R}^n$, i.e., $f : \mathbf{R}^n \to \mathbf{R}$ is continuously differentiable and for any $x, y \in \mathbf{R}^n$ and any $\alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$. Suppose that the gradient of f is Lipschitz continuous, i.e., there exists a constant $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2.$$

Prove the following inequalities:

(i). $f(y) \leq f(x) + (\nabla f(x))^T (y-x) + \frac{L}{2}\|y-x\|_2^2, \quad \forall x, y \in \mathbf{R}^n$;

(ii). $f(y) \geq f(x) + (\nabla f(x))^T (y-x) + \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|_2^2, \quad \forall x, y \in \mathbf{R}^n$;

(iii). $\frac{1}{L}\|\nabla f(y) - \nabla f(x)\|_2^2 \leq (\nabla f(y) - \nabla f(x))^T (y-x), \quad \forall x, y \in \mathbf{R}^n$.

5) (30 pointes) Consider the following problems.

(i) Determine the order of Störmer's method,

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f(t_{n+1}, y_{n+1}), \qquad n \geqslant 0,$$

for solving the second order system of ODE's

$$y'' = f(t, y), \qquad t \geqslant 0,$$

with the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$.

(ii) Using the second order central differences in space and Störmer's method in time, construct a scheme to solve the wave equation,

$$u_{tt} = u_{xx}.$$

(iii) Determine the condition for its stability.