S.-T. Yau College Student Mathematics Contests 2022

Algebra and Number Theory

Solve every problem.

Problem 1.

(a) Let $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in R[x]$ be a polynomial over an integral domain R. Let K denote the fraction field of R. Suppose $a/b \in K$ is a root of $p(x)$, where $a, b \in R$ and are relatively prime. Then, show that $a | a_0$ and $b | a_n$.

(b) Prove that $\mathbf{Q}(\sqrt{2}, \sqrt{3}) = \mathbf{Q}(\sqrt{2} + \sqrt{3})$.

Problem 2. Let R be an integral domain with the fraction field K. An R-module P is projective if there is an R-module Q such that $P \bigoplus Q \cong F$ for some free R-module F. A fractional ideal A is an R-submodule of K such that $A = d^{-1} I$ for some ideal I of R and a nonzero element $d \in R$. A fractional ideal A is called invertible if $AB = R$ for some fractional ideal B.

Show that an invertible fractional ideal A is a projective R-module.

Problem 3. Give a *direct* proof that the Lie algebra $\mathfrak{sl}(4, \mathbf{C})$ is isomorphic to the Lie algebra $\mathfrak{so}(6, \mathbf{C})$. (You should construct a Lie algebra homomorphism and prove that it is an ismorphism; you should not use Dynkin diagrams or the classification theory of simple Lie algebras.)

Problem 4. Let $A = \mathcal{O}_K$ be the ring of integers of a number field K. Given a nonzero ideal $\mathfrak{a} \subset A$ and an arbitrary nonzero element $a \in \mathfrak{a}$, show that there exists $b \in \mathfrak{a}$ such that a and b generate $\mathfrak{a}$ (in particular, every ideal is 2-generated).

Problem 5. Let p be a prime number and ζ_p be a primitive p-th root of unity. Let $K = \mathbf{Q}(\zeta_p)$.

(a) Show that $\Phi_p = \sum_{i=0}^{p-1} X^i$ is the minimal polynomial of ζ_p over $\mathbf{Q}$.

(b) Compute the trace $\mathrm{Tr}_{K/\mathbf{Q}}(1 - \zeta_p)$ and the norm $\mathcal{N}_{K/\mathbf{Q}}(1 - \zeta_p)$.

(c) Show that $(1 - \zeta_p)\mathcal{O}_K \cap \mathbf{Z} = p\mathbf{Z}$ and deduce that for all $y \in \mathcal{O}_K$, we have

$$\mathrm{Tr}_{K/\mathbf{Q}}(y(1 - \zeta_p)) \in p\mathbf{Z}.$$

(d) Determine explicitly the ring of integers of K.

Problem 6. Let $\theta \in \overline{\mathbf{Q}}$ be a root of the polynomial $f(X) = X^3 + 12X^2 + 8X + 1$. Let $K = \mathbf{Q}(\theta)$.

(a) Let $g(X) = X^3 + pX + q \in \mathbf{Z}[X]$. Compute the discriminant $\mathrm{disc}(g)$ of $g(X)$ in terms of p, q.

(b) Show that $f(X)$ is irreducible over $\mathbf{Q}$.

(c) Compute the discriminant $d_K(1, \theta, \theta^2)$. Please provide necessary details.

(d) For any arbitrary number field F of degree n, let $a_1, a_2, \ldots, a_n \in \mathcal{O}_F$. Find and verify a sufficient condition in terms of the discriminant $d_F(a_1, \ldots, a_n)$ that the $a_1, \ldots, a_n$ form an integral basis of F.

(e) Write down an explicit integral basis of K in terms of θ by using the above sufficient condition you have found. Please justify your arguments.