S.-T. Yau College Student Mathematics Contests 2023

Analysis and Differential Equations

Solve every problem.

1. Let $(M_n, \|\cdot\|)$ be the Banach space formed by $n \times n$ real matrices, equipped with Hilbert-Schmidt norm. Prove:

   (1). If $\gamma : \mathbb{R} \to M_n$ is a C^1-function with $\gamma(0) = \mathbb{I}$ (the identity matrix) and $\dot{\gamma}(0) = A \in M_n$, then for $\forall t \in \mathbb{R}$, the sequence $\{\gamma^n(t/n)\}_{n=1}^{\infty}$ converges to $\exp(tA)$; In particular, for $\forall A, B \in M_n$, $e^{t(A+B)} = \lim_{n\to\infty}(e^{\frac{t}{n}A}e^{\frac{t}{n}B})^n$;

   (2). For $\forall A, B \in M_n$, if $e^{t(A+B)} = e^{tA}e^{tB}$, $\forall t \in \mathbb{R}$, then $[A, B] := AB - BA = 0$.

2. Let $f(t, x, y)$ be a C^1 function on $[0, 1] \times \mathbb{R}^2$. Let φ be a solution of the second-order ordinary differential equation $(*)\frac{d^2x}{dt^2} = f(t, x, \frac{d}{dt}x)$, $t \in [0, 1]$ such that $\varphi(0) = a$, $\varphi(1) = b$ where a, b are given real numbers. Suppose $\frac{\partial f}{\partial x}(t, x, y) > 0$ for all $(t, x, y) \in [0, 1] \times \mathbb{R}^2$.

   Prove: if $|\beta - b|$, $\beta \in \mathbb{R}$, is sufficiently small, then there exists a solution ψ of $(*)$ such that $\psi(0) = a$, $\psi(1) = \beta$.

3. A function $\varphi : \mathbb{R}^n \to \mathbb{C}$ is called positive definite if for all $k \in \mathbb{N}$, $y_j \in \mathbb{R}^n$, $c_j \in \mathbb{C}$, $j = 1, \dots, k$, one have $\sum_{i,j=1}^{k} c_i\bar{c}_j\varphi(y_i - y_j) \geqslant 0$. If φ is a measurable positive definite function on $\mathbb{R}^n$. Prove:

   (1) $\varphi(-y) = \overline{\varphi(y)}$ and $|\varphi(y)| \leqslant \varphi(0)$.

   (2) For every Lebesgue integrable nonnegative function f on $\mathbb{R}^n$, one have
   $$\int_{\mathbb{R}^n} \varphi(x - y)f(x)f(y)dxdy \geqslant 0.$$

   (3) If the function f is also even, then
   $$\int_{\mathbb{R}^n} \varphi(x)f * f(x)dx \geqslant 0,$$
   where $*$ stands for convolution.

   (4) For all $\alpha > 0$, we have
   $$\int_{\mathbb{R}^n} \varphi(x)\exp(-\alpha|x|^2)dx \geqslant 0.$$

4. For a 2π-periodic function $x \in L^2([0, 2\pi])$, let $x_n(t) = x(nt)$, $n = 1, 2, \dots$.

   Prove that $\{x_n\}$ converges weakly in $L^2([0, 2\pi])$ and find the limit.

5. Let f be an entire function on $\mathbb{C}$ and there exists a constant $C > 0$ such that $f(z)| \leqslant C\sqrt{|z|}|\cos(z)|$ for all $z \in \mathbb{C}$. Prove that f is identically zero.

6. Let $u(t, x)$ satisfy the following equation,

$$u_t + \sum_{i=1}^{n} \psi_i(t, x, u) u_{x_i} = \mu \Delta u; \; u(0, x) = u_0,$$

   where $u_0 \in C^2(\mathbb{R}^n)$, ψ_i, $i = 1, \dots, n$ are of bounded C^2-norm, $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ is the Laplacian in $\mathbb{R}^n$ and $\mu > 0$. Assume $|u_0| \leqslant e^{\Phi/\mu}$ where Φ is bounded above and has Lipschitz constant 1. Assume that $(\sum_{i=1}^n |\Psi_i(t, x, u)|^2)^{1/2} \leqslant A$, where A is a postive constant.

   Prove that there exists a constant C depending only on n that

$$|u(t, x)| \leqslant e^{Ct} e^{((A+1)t + \Phi(x) + 2)/\mu}, \quad \forall t \geqslant 0.$$