S.-T. Yau College Student Mathematics Contests 2010

Applied Math., Computational Math., Probability and Statistics

Individual

(Please select 5 problems to solve)

1. Let $Z_1, \cdots, Z_n$ be i.i.d. random variables with $Z_i \sim N(\mu, \sigma^2)$. Find

$$E(\sum_{i=1}^{n} Z_i | Z_1 - Z_2 + Z_3).$$

2. Let $X_1, \cdots, X_n$ be pairwise independent. Further, assume that $EX_i = 1 + i^{-1}$ and that $\max_{1\le i\le n} E|X_i|^{1+\epsilon} < \infty$ for some $\epsilon > 0$. Show that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} 1.$$

3. Let $Z_1, \cdots, Z_6$ be i.i.d. random variables with $Z_i \sim N(0,1)$. Set

$$U^2 = \frac{(Z_1Z_2 + Z_3Z_4 + Z_5Z_6)^2}{Z_2^2 + Z_4^2 + Z_6^2}, \quad V^2 = \frac{U^2(Z_2^2 + Z_4^2)}{U^2 + Z_6^2}.$$

Find and identify the densities of U^2 and V^2.

4. Suppose that three characteristics in a large propulation can be observed according to the following frequencies

$$p_1 = \theta^3, \quad p_2 = 3\theta(1-\theta), \quad p_3 = (1-\theta)^3,$$

where $\theta \in (0,1)$. Let $N_j,\ j = 1,2,3$ be the observed frequencies of characteristic j in a random sample of size n.

(a) Construct the approximate level $(1-\alpha)$ maximum likelihood confidence set for θ.

(b) Derive the asymptotic distribution for the frequency substitution estimator $\hat{\theta}_2 = 1 - (N_3/n)^{1/3}$.

5. (1) Suppose

$$S = \begin{bmatrix} \sigma & \mathbf{u}^T \\ 0 & S_c \end{bmatrix}, \quad T = \begin{bmatrix} \tau & \mathbf{v}^T \\ 0 & T_c \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \beta \\ \mathbf{b}_c \end{bmatrix},$$

where σ, τ and β are scalars, S_c and T_c are n-by-n matrices, and $\mathbf{b}_c$ is an n-vector. Show that if there exists a vector $\mathbf{x}_c$ such that

$$(S_cT_c - \lambda I)\mathbf{x}_c = \mathbf{b}_c$$

and $\mathbf{w}_c = T_c\mathbf{x}_c$ is available, then

$$\mathbf{x} = \begin{bmatrix} \gamma \\ \mathbf{x}_c \end{bmatrix}, \quad \gamma = \frac{\beta - \sigma\mathbf{v}^T\mathbf{x}_c - \mathbf{u}^T\mathbf{w}_c}{\sigma\tau - \lambda}$$

solves $(ST - \lambda I)\mathbf{x} = \mathbf{b}$.

(2) Hence or otherwise, derive an $O(n^2)$ algorithm for solving the linear system $(U_1U_2 - \lambda I)\mathbf{x} = \mathbf{b}$ where U_1 and U_2 are n-by-n upper triangular matrices, and $(U_1U_2 - \lambda I)$ is nonsingular. Please write down your algorithm and prove that it is indeed of $O(n^2)$ complexity.

(3) Hence or otherwise, derive an $O(pn^2)$ algorithm for solving the linear system $(U_1U_2 \cdots U_p - \lambda I)\mathbf{x} = \mathbf{b}$ where $\{U_i\}_{i=1}^p$ are all n-by-n upper triangular matrices, and $(U_1U_2 \cdots U_p - \lambda I)$ is nonsingular. Please write down your algorithm and prove that it is indeed of $O(pn^2)$ complexity.

6. (1) Let $A \in \mathbb{R}^{m\times n}$, i.e. A is an m-by-n real matrix. Show that there exists an m-by-m orthogonal matrix U and an n-by-n orthogonal matrix V such that

$$U^TAV = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p),$$

where $p = \min\{m, n\}$ and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0.$$

(2) Let rank$(A)= r$. Show that for any positive integer $k < r$,

$$\min_{\mathrm{rank}(B)=k} \|A - B\|_2 = \sigma_{k+1}.$$

(*Hint:* Consider the matrix $A_k = \sum_{i=1}^{k} \sigma_i\mathbf{u}_i\mathbf{v}_i^T$, where $\mathbf{u}_i$ and $\mathbf{v}_i$ are columns of U and V respectively.)