S.-T. Yau College Student Mathematics Contests 2014

Probability and Statistics Problems

Team

Please solve the following 5 problems.

Problem 1. Suppose that X_n converges to X in distribution and Y_n converges to a constant c in distribution. Show that

(a) Y_n converges to c in probability;

(b) $X_n Y_n$ converges to cX in distribution.

Problem 2. Let X and Y be two random variables with $|Y| > 0$, a.s.. Let $Z = X/Y$.

(a) Assume the distribution function of (X, Y) has the density $p(x, y)$. What is the density function of Z?

(b) Assume X and Y are independent and X is $N(0, 1)$ distributed, Y has the uniform distribution on $(0, 1)$. Give the density function of Z.

Problem 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$, and $\Gamma \in \mathcal{F}$. Prove that the following properties are equivalent:

(i) Γ is independent of $\mathcal{G}$ under P,

(ii) for every probability Q on $(\Omega, \mathcal{F})$, equivalent to P, with dQ/dP being $\mathcal{G}$ measurable, we have $Q(\Gamma) = P(\Gamma)$.

(b) Let X, Y, Z be random variables and Y is integrable. Show that if (X, Y) and Z are independent, then $E[Y|X, Z] = E[Y|X]$.

Problem 4. Let $X_1, ..., X_n$ be i.i.d. $N(0, \sigma^2)$, and let M be the mean of $|X_1|, ..., |X_n|$.

1. Find $c \in R$ so that $\hat{\sigma} = cM$ is a consistent estimator of σ.
2. Determine the limiting distribution for $\sqrt{n}(\hat{\sigma} - \sigma)$.
3. Identify an approximate $(1 - \alpha)\%$ confidence interval for σ.
4. Is $\hat{\sigma} = cM$ asymptotically efficient? Please justify your answer.

Problem 5. The shifted exponential distribution has the density function

$$f(y;\phi,\theta) = 1/\theta \exp\{-(u-\phi)/\theta\}, \qquad y > \phi, \theta > 0.$$

Let $Y_1, \ldots, Y_n$ be a random sample from this distribution. Find the maximum likelihood estimator (MLE) of ϕ and θ and the limiting distribution of the MLE.
You may use the following Rényi representation of the order statistics: Let $E_1, \ldots, E_n$, be a random sample from the standard exponential distribution (i.e., the above distribution with $\phi = 0, \theta = 1$). Let $E_{(r)}$ denote the r-th order statistics. According to the Rényi representation,

$$E_{(r)} \stackrel{D}{=} \sum_{j=1}^{r} \frac{E_j}{n+1-j}, \qquad r = 1, \ldots, n.$$

Here, the symbol $\stackrel{D}{=}$ denotes equal in distribution.