S.-T. Yau College Student Mathematics Contests 2019

Algebra and Number Theory

Individual (5 problems)

1) Let G be a finite group. Assume that for any representation V of G over a field of characteristic zero, the character χ_V takes value in $\mathbb{Q}$. Assume g is an element in G such that $g^{2019} = 1$.

   Prove that g and g^{19} are conjugate in G.

2) Let p be a prime number, and let $\mathbb{F}_p$ be the finite field with p elements. Let $F = \mathbb{F}_p(t)$ be the field of rational functions over $\mathbb{F}_p$. Consider all subfields C of F such that F/C is a finite Galois extension.

   1. Show that among such subfields, there is a smallest one C_0, i.e., C_0 is contained in any other C.
   2. What is the degree of F/C_0 ?

3) Let $R \subset R'$ be an integral extension of commutative rings. Let $\mathfrak{p}'$ be a prime ideal of R'. Prove that $\mathfrak{p}'$ is a maximal ideal of R' if and only if $\mathfrak{p}' \cap R$ is a maximal ideal of R.

4)
   1. Prove that $\mathrm{GL}_n(\mathbb{C})$ is path-connected.
   2. Let
   $$X = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A^m = \mathrm{Id}\},$$
   Describe the path-connected component of X and prove your answer.

5) The Fibonacci sequence is defined by

   $$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n.$$

   Let p be a prime number.

   1. Show that if $p \equiv 1, 4 \pmod 5$, then p divides F_{p-1}.
   2. Let $\mathbb{F}_{p^2}$ be the finite field of p^2 elements. Show that the norm map $N : \mathbb{F}_{p^2}^\times \to \mathbb{F}_p^\times$ is surjective, and deduce the cardinality of the kernel of N.
   3. Show that if $p \equiv 2, 3 \pmod 5$, then p divides F_{p+1}.