S.-T. Yau College Student Mathematics Contests 2011

Algebra, Number Theory and Combinatorics

Team

9:00–12:00 pm, July 9, 2011

(Please select 5 problems to solve)

For the following problems, every example and statement must be backed up by proof. Examples and statements without proof will receive no-credit.

1. Let F be a field and $\bar{F}$ the algebraic closure of F. Let $f(x, y)$ and $g(x, y)$ be polynomials in $F[x, y]$ such that g.c.d.$(f, g) = 1$ in $F[x, y]$. Show that there are only finitely many $(a, b) \in \bar{F}^{\times 2}$ such that $f(a, b) = g(a, b) = 0$. Can you generalize this to the cases of more than two-variables?

2. Let D be a PID, and D^n the free module of rank n over D. Then any submodule of D^n is a free module of rank $m \leq n$.

3. Identify pairs of integers $n \neq m \in \mathbb{Z}_+$ such that the quotient rings $\mathbb{Z}[x, y]/(x^2 - y^n) \cong \mathbb{Z}[x, y]/(x^2 - y^m)$; and identify pairs of integers $n \neq m \in \mathbb{Z}_+$ such that $\mathbb{Z}[x, y]/(x^2 - y^n) \ncong \mathbb{Z}[x, y]/(x^2 - y^m)$.

4. Is it possible to find an integer $n > 1$ such that the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is an integer?

5. Recall that $\mathbb{F}_7$ is the finite field with 7 elements, and $GL_3(\mathbb{F}_7)$ is the group of all invertible 3×3 matrices with entries in $\mathbb{F}_7$.

(a) Find a 7-Sylow subgroup P_7 of $GL_3(\mathbb{F}_7)$.

(b) Determine the normalizer subgroup N of the 7-Sylow subgroup you found in (a).

(c) Find a 2-Sylow subgroup of $GL_3(\mathbb{F}_7)$.

6. For a ring R, let $SL_2(R)$ denote the group of invertible 2×2 matrices. Show that $SL_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. What about $SL_2(\mathbb{R})$?