S.-T. Yau College Student Mathematics Contests 2011

Algebra, Number Theory and Combinatorics

Individual

2:30–5:00 pm, July 10, 2011
(Please select 5 problems to solve)

For the following problems, every example and statement must be backed up by proof. Examples and statements without proof will receive no-credit.

1. Let $K = \mathbb{Q}(\sqrt{-3})$, an imaginary quadratic field.

(a) Does there exists a finite Galois extension $L/\mathbb{Q}$ which contains K such that $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$? (Here S_3 is the symmetric group in 3 letters.)

(b) Does there exists a finite Galois extension $L/\mathbb{Q}$ which contains K such that $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$?

(c) Does there exists a finite Galois extension $L/\mathbb{Q}$ which contains K such that $\mathrm{Gal}(L/\mathbb{Q}) \cong Q$? Here Q is the quaternion group with 8 elements $\{\pm 1, \pm i, \pm j, \pm k\}$, a finite subgroup of the group of units $\mathbb{H}^\times$ of the ring $\mathbb{H}$ of all Hamiltonian quaternions.

2. Let f be a two-dimensional (complex) representation of a finite group G such that 1 is an eigenvalue of $f(\sigma)$ for every $\sigma \in G$. Prove that f is a direct sum of two one-dimensional representations of G

3. Let $F \subset \mathbb{R}$ be the subset of all real numbers that are roots of monic polynomials $f(X) \in \mathbb{Q}[X]$.

(1) Show that F is a field.

(2) Show that the only field automorphisms of F are the identity automorphism $\alpha(x) = x$ for all $x \in F$.

4. Let V be a finite-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear transformation such that

(1) the minimal polynomial of T is irreducible;

(2) there exists a vector $v \in V$ such that $\{T^i v \mid i \geq 0\}$ spans V.

Show that V contains no non-trivial proper T-invariant subspace.

5. Given a commutative diagram

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

of Abelian groups, such that (i) both rows are exact sequences and (ii) every vertical map, except the middle one, is an isomorphism. Show that the middle map $C \to C'$ is also an isomorphism.

6. Prove that a group of order 150 is not simple.