S.-T. Yau College Student Mathematics Contests 2019

Geometry and Topology

Team (5 problems)

1) Is TS^2 diffeomorphic to $S^2 \times \mathbb{R}^2$? Verify your answer. Here TS^2 is the total space of the tangent bundle of S^2.

2) Solve the problem which Russell Crowe assigns to his students in the movie "A beautiful mind" (2001):

$$V = \{F : \mathbb{R}^3 \backslash X \to \mathbb{R}^3 \; s.t. \; \nabla \times F = 0\}$$
$$W = \{F = \nabla g\}$$
$$\dim (V/W) = ?$$

First give the general answer for any closed $X \subset \mathbb{R}^3$, and then specialize it to (a) $X = \{x = y = z = 0\}$, (b) $X = \{x = y = 0\}$ and (c) $X = \{x = 0\}$.

3) Let $T^2 = S^1 \times S^1$ be the 2-torus with the standard orientation, and let $F : T^2 \to T^2$ be a smooth map of degree 1 such that $F \circ F = \mathrm{Id}$ and F has no fixed points. Prove that the induced map $F^* : H^1(T^2) \to H^1(T^2)$ is the identity.

4) Let $U(n)$ be the group of $n \times n$ unitary matrices, and $O(n)$ be the group of $n \times n$ orthogonal matrices. Let $SU(n) = \{A \in U(n) | \det A = 1\}$ be the special unitary group and $SO(n) = \{A \in O(n) | \det A = 1\}$ be the special orthogonal group. All $U(n), SU(n), O(n), SO(n)$ are Lie groups with natural manifold structures.

   (a) Compute the dimensions of $SU(n)$ and $SO(n)$.

   (b) Compute the fundamental groups of $SU(n)$ and $SO(n)$ $(n \geqslant 2)$.

5) Let (M, g) be a compact Riemannian manifold and R be its Riemannian curvature tensor. (M, g) will be called *weakly negative* if for any point $p \in M$ and for any nonzero vector field $X \in T_pM$, there exists a nonzero vector field $Y \in T_pM$ such that $R(X, Y, Y, X) < 0$.

   (a) Let X be a Killing vector field and $f = \frac{1}{2}g(X, X) = \frac{1}{2}|X|^2$. Show that for any vector field V

   $$(\mathrm{Hess} f)\,(V, V) = g(\nabla_V X, \nabla_V X) - R(V, X, X, V).$$

   Here the Hessian of f is $(\mathrm{Hess} f)\,(Y, Z) := g(\nabla_Y \mathrm{grad}(f), Z)$ for any vector fields Y, Z, where $\mathrm{grad}(f)$ is the gradient vector of f.

   (b) Prove that if (M, g) is weakly negative, then there are no nontrivial Killing vector fields.