S.-T. Yau College Student Mathematics Contests 2010

Algebra, Number Theory and Combinatorics

Individual

(Please select 5 problems to solve)

1. Let V be a finite dimensional complex vector space. Let A, B be two linear endomorphisms of V satisfying $AB - BA = B$. Prove that there is a common eigenvector for A and B.

2. Let $M_2(\mathbb{R})$ be the ring of 2×2 matrices with real entries. Its group of multiplicative units is $GL_2(\mathbb{R})$, consisting of invertible matrices in $M_2(\mathbb{R})$.

(a) Find an injective homomorphism from the field $\mathbb{C}$ of complex numbers into the ring $M_2(\mathbb{R})$.

(b) Show that if ϕ_1 and ϕ_2 are two such homomorphisms, then there exists a $g \in GL_2(\mathbb{R})$ such that $\phi_2(x) = g\phi_1(x)g^{-1}$ for all $x \in \mathbb{C}$.

(c) Let h be an element in $GL_2(\mathbb{R})$ whose characteristic polynomial $f(x)$ is irreducible over $\mathbb{R}$. Let $F \subset M_2(\mathbb{R})$ be the subring generated by h and $a \cdot I$ for all $a \in \mathbb{R}$, where I is the idenity matrix. Show that F is isomorphic to $\mathbb{C}$.

(d) Let h' be any element in $GL_2(\mathbb{R})$ with the same characteristic polynomial $f(x)$ as h in (c). Show that h and h' are conjugate in $GL_2(\mathbb{R})$.

(e) If $f(x)$ in (c) and (d) is reducible over $\mathbb{R}$, will the same conclusion on h and h' hold? Give reasons.

3. Let G be a non-abelian finite group. Let $c(G)$ be the number of conjugacy classes in G Define $\overline{c}(G) := c(G)/|G|$, $(|G| = \mathrm{Card}(G))$.

(a) Prove that $\overline{c}(G) \leq \frac{5}{8}$.

(b) Is there a finite group H with $\overline{c}(H) = \frac{5}{8}$?

(c) (open ended question) Suppose that there exists a prime number p and an element $x \in G$ such that the cardinality of the conjugacy class of x is divisible by p. Find a good/sharp upper bound for $\overline{c}(G)$.

4. Let F be a splitting field over $\mathbb{Q}$ the polynomial $x^8 - 5 \in \mathbb{Q}[x]$. Recall that F is the subfield of $\mathbb{C}$ generated by all roots of this polynomial.

(a) Find the degree $[F : \mathbb{Q}]$ of the number field F.
(b) Determine the Galois group $\mathrm{Gal}(F/\mathbb{Q})$.

5. Let $T \subset \mathbb{N}_{>0}$ be a finite set of positive integers. For each integer $n > 0$, define a_n to be the number of all finite sequences $(t_1, \ldots, t_m)$ with $m \leq n$, $t_i \in T$ for all $i = 1, \ldots, m$ and $t_1 + \ldots + t_m = n$. Prove that the infinite series

$$1 + \sum_{n \geq 1} a_n \, z^n \in \mathbb{C}[[z]]$$

is a *rational* function in z, and find this rational function.

6. Describe all the irreducible complex representations of the group S_4 (the symmetric group on four letters).