S.-T. Yau College Student Mathematics Contests 2021

Computational and Applied Mathematics

Solve every problem.

Problem 1.

(a) Show that

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1, 1],$$

is a polynomial of degree n with extrema at

$$x_k = \cos\left(k\frac{\pi}{n}\right), \quad k = 0, 1, \ldots, n$$

and leading coefficient 2^{n-1}.

(b) Show that if $f \in C^{n+1}[-1, 1]$ and if $P(x)$ is the polynomial with degree at most n that interpolates f at x_k, $k = 0, 1, \ldots, n$ then

$$\|f(x) - P(x)\|_\infty \le \frac{1}{2^{n-1}(n+1)!}\left\|f^{n+1}\right\|_\infty .$$

Solution:

(a)

$$\cos(n \arccos(\cos(k\pi/n))) = \cos(k\pi) = (-1)^k \quad \text{for } k = 0, 1, \ldots, n.$$

To show the degree of T_n, we use induction. $T_0(x) = 1$ and $T_1(x) = x$.
Induction hypothesis: T_k is a polynomial of degree k and leading coefficient 2^{k-1} for $k \le n$.
Note that

$$\begin{aligned} T_{n+1}(x) + T_{n-1}(x) &= \cos\left((n+1) \arccos x\right) + \left((n-1) \arccos x\right) \\ &= 2x \cos(n \arccos x) \\ &= 2xT_n(x). \end{aligned}$$

Now suppose T_n is of degree n and with leading coefficient 2^{n-1}. From the above calculation,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

which shows that T_{n+1} is of degree $n+1$ and has leading coefficient 2^n. This completes the proof to part **(a)**.

(b) Essentially, this boils down to proving the following,

$$\max_{[-1,1]} |(x - x_0)(x - x_1)\cdots(x - x_n)| \le \frac{1}{2^{n-1}}.$$

To show this, we define a new Chebyshev-like polynomial. Define

$$Q_n(x) := \sin(n \arccos x)\sqrt{1 - x^2}, \quad n = 1, 2, \ldots$$

Claim: Q_n is of degree $n+1$ with leading coefficient -2^{n-1}.
We prove this claim by induction.
$Q_1(x) = 1 - x^2$ and $Q_2(x) = 2x \sin(\arccos x)\sqrt{1 - x^2} = 2x(1 - x^2)$. This satisfies the claim and serves as the

base case.
Induction hypothesis: Q_k is a polynomial of degree $k+1$ and leading coefficient -2^{k-1} for $k \le n$.
Note,

$$\begin{aligned} Q_{n+1}(x) - Q_{n-1}(x) &= [\sin((n+1)\arccos x) - \sin((n-1)\arccos x)]\sqrt{1-x^2} \\ &= 2\cos(n\arccos x)\sin(\arccos x)\sqrt{1-x^2} \\ &= 2(1-x^2)T_n(x). \end{aligned}$$

Therefore,

$$Q_{n+1}(x) = 2(1-x^2)T_n(x) + Q_{n-1}(x).$$

Using part **(a)** and the induction hypothesis, Q_{n+1} is a polynomial of degree $n+2$ and leading coefficient -2^n. This completes the proof to claim.

Also note that $x_0, x_1, \ldots, x_n$ are the roots of Q_n. As a result,

$$Q_n = -2^{n-1}(x-x_0)(x-x_1)\ldots(x-x_n).$$

Since $\max_{[-1,1]} Q_n = 1$, the result follows.

Problem 2. Let $S(x)$ be a cubic spline with knots $\{t_i\}_{i=0}^n$. If it is determined that $S(x)$ is linear over $[t_1, t_2]$ and $[t_3, t_4]$. Prove that $S(x)$ is also linear over $[t_2, t_3]$.

Solution: First define $p : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} p(x) =& \frac{S''(t_3)(x-t_2)^3}{6(t_3-t_2)} + \frac{S''(t_2)(t_3-x)^3}{6(t_3-t_2)} + \left[\frac{S(t_3)}{t_3-t_2} - \frac{S''(t_3)(t_3-t_2)}{6}\right](x-t_2) \\ &+ \left[\frac{S(t_2)}{t_3-t_2} - \frac{S''(t_2)(t_3-t_2)}{6}\right](t_3-x). \end{aligned}$$

We claim that $p = S$ in $[t_2, t_3]$. Since $\deg(p) = \deg(S) = 3$, we will be done if we show p and S match at four distinct constraints. Observe

$$p(t_2) = 0 + \frac{S''(t_2)(t_3-t_2)^2}{6} + 0 + \left[\frac{S(t_2)}{t_3-t_2} - \frac{S''(t_2)(t_3-t_2)}{6}\right](t_3-t_2) = S(t_2).$$

In a similar fashion, we also have

$$p(t_3) = S(t_3).$$

Moreover,

$$p''(x) = S''(t_3)\frac{x-t_2}{t_3-t_2} + S''(t_2)\frac{x-t_3}{t_2-t_3},$$

which is the Lagrange interpolating polynomial between $S''(t_2)$ and $S''(t_3)$, *i.e.*,

$$p''(t_i) = S''(t_i),$$

for $i = 2, 3$. This shows four degrees of freedom for which p matches S, and so we conclude $p = S$ in $[t_2, t_3]$.

We use the fact $p = S$ to show S is linear over $[t_2, t_3]$. Because S is linear over $[t_1, t_2]$ and $[t_3, t_4]$, we have $S''(t_2) = S''(t_3) = 0$. This implies for $[t_2, t_3]$,

$$S(x) = p(x) = 0 + 0 + \left[\frac{S(t_3)}{t_3-t_2} - 0\right](x-t_2) + \left[\frac{S(t_2)}{t_3-t_2} - 0\right](t_3-x) = S(t_3)\frac{x-t_2}{x_3-x_2} + S(t_2)\frac{x-t_3}{x_2-x_3},$$

i.e., $\deg(S) = 1$. Thus S is linear over $[t_2, t_3]$.

Problem 3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - \cos x$.

(a) Prove that the equation $f(x) = 0$ has a unique solution $x^* \in \mathbb{R}$ that lies in the interval $(\frac{1}{4}, \frac{1}{2})$.

(b) Prove that the sequence defined by the fixed point iteration

$$\begin{aligned} &x_0, \\ &x_n = \tfrac{1}{2}\cos x_{n-1}, \quad n = 1, 2, \dots \end{aligned}$$

converges to x^* with any initial guess x_0.

(c) For the fixed point iteration in **(b)** with $x_0 = \frac{\pi}{6}$, determine an n that guarantees $|x_n - x^*| < \frac{1}{2} \times 10^{-8}$. For the fixed point iteration in **(b)** with $x_0 = 20$, determine an n that guarantees $|x_n - x^*| < \frac{1}{4}$.

Solution:

(a) $f(\frac{1}{4}) = \frac{1}{2} - \cos\frac{1}{4} < \frac{1}{2} - \cos\frac{\pi}{4} < 0$. Also, $f(\frac{1}{2}) = 1 - \cos\frac{1}{2} > 0$. Therefore, by the Intermediate Value Theorem, there exists a root in the said interval. However, since $f' = 2 + \sin x$, the function is strictly increasing and the root is unique.

(b) Set $\phi(x) := \frac{1}{2}\cos x$. The iteration scheme is $x_{n+1} = \phi(x_n)$.

$$\begin{aligned} |x_{n+1} - x^*| &= |\phi(x_n) - x^*| \\ &= |\phi(x_n) - \phi(x^*)| \\ &= |\phi'(\xi)| \cdot |x_n - x^*| \\ &\le \tfrac{1}{2}\,|x_n - x^*|\,. \end{aligned}$$

Therefore,

$$|x_n - x^*| \le \frac{1}{2^n}|x_0 - x^*|,$$

which converges.

(c) For $x_0 = \frac{\pi}{6}$, using part **(b)**, a necessary condition to ensure the required bound is,

$$\frac{\left|\frac{\pi}{6} - \frac{1}{4}\right|}{2^n} < \frac{10^{-8}}{2},$$

which is,

$$n > 1 + \log_2\left[10^8\left(\frac{\pi}{6} - \frac{1}{4}\right)\right] \approx 25.71.$$

Hence, $n = 26$ would suffice. For $x_0 = 20$,

$$\frac{20 - \frac{1}{4}}{2^n} < \frac{1}{4}.$$

So,

$$n > 2 + \log_2 19.75 > 6.$$

Hence, $n = 7$ would suffice.

Problem 4. Let matrix $\mathbf{A} \in \mathbf{R^{m\times n}}$ with $m \ge n$ and $r = \operatorname{rank}(\mathbf{A}) < n$, and assume A has the following SVD decomposition

$$\mathbf{A} = [\mathbf{U_1}, \mathbf{U_2}]\begin{bmatrix} \boldsymbol{\Sigma_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}[\mathbf{V}_1, \mathbf{V}_2]^T = \mathbf{U}_1\Sigma_1\mathbf{V}_1^T,$$

where Σ_1 is $r \times r$ nonsingular and $\mathbf{U}_1$ and $\mathbf{V}_1$ have r columns. Let $\sigma = \sigma_{\min}(\Sigma_1)$, the smallest nonzero singular value of $\mathbf{A}$. Consider the following least square problem, for some $\mathbf{b} \in \mathbf{R^m}$,

$$\min_{\mathbf{x}\in\mathbf{R^n}} \|\mathbf{Ax} - \mathbf{b}\|_2\,.$$

(a) Show that all solutions $\mathbf{x}$ can be written as

$$\mathbf{x} = \mathbf{V_1}\Sigma_1^{-1}\mathbf{U_1^T b} + \mathbf{V_2 z_2},$$

with $\mathbf{z_2}$ an arbitrary vector.

(b) Show that the solution $\mathbf{x}$ has minimal norm $\|\mathbf{x}\|_2$ precisely when $\mathbf{z_2} = \mathbf{0}$, and in which case,

$$\|\mathbf{x}\|_2 \leq \frac{\|\mathbf{b}\|_2}{\sigma}.$$

Solution:

(a) Set $\mathbf{U} = [\mathbf{U_1}, \mathbf{U_2}], \Sigma = \begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, and $\mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2]^T$, then the SVD decomposition of $\mathbf{A}$ is $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V^T}$, where $\mathbf{U_{m\times m}}, \mathbf{V_{n\times n}}$ are orthogonal matrices such that

- $\mathbf{U^T} = \mathbf{U}^{-1}$ and $\mathbf{V^T} = \mathbf{V^{-1}}$
- $\mathbf{U}$ and $\mathbf{V}$ are l_2-norm preserving.

As a consequence

$$\|\mathbf{Ax} - \mathbf{b}\|_2 = \left\|\mathbf{U}\Sigma\mathbf{V}^T\mathbf{x} - \mathbf{U}\mathbf{U^T b}\right\|_2 = \left\|\Sigma\mathbf{V}^T\mathbf{x} - \mathbf{U^T b}\right\|_2.$$

Let $\mathbf{z} = \mathbf{V^T x} = (\mathbf{z_1}, \mathbf{z_2})^{\mathbf{T}}$ and $\mathbf{c} = \mathbf{U^T b} = (\mathbf{c_1}, \mathbf{c_2})^{\mathbf{T}}$. Then

$$\|\mathbf{Ax} - \mathbf{b}\|_2 = \left\|\begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix} - \begin{bmatrix} \mathbf{c_1} \\ \mathbf{c_2} \end{bmatrix}\right\|_2 = \left\|\begin{bmatrix} \Sigma_1\mathbf{z_1} - \mathbf{c_1} \\ \mathbf{c_2} \end{bmatrix}\right\|_2.$$

The l_2-norm is minimized when the vector $\mathbf{z}$ is chosen with $\mathbf{z_1} = \Sigma_1^{-1}\mathbf{c_1}$, $\mathbf{z_2}$ arbitrary. Then

$$\mathbf{x} = \mathbf{Vz} = \mathbf{V}\begin{bmatrix} \Sigma_1^{-1}\mathbf{c_1} \\ \mathbf{z_2} \end{bmatrix} = (\mathbf{V_1}, \mathbf{V_2})\begin{bmatrix} \Sigma_1^{-1}\mathbf{U_1^T b} \\ \mathbf{z_2} \end{bmatrix} = \mathbf{V_1}\Sigma_1^{-1}\mathbf{U_1^T b} + \mathbf{V_2 z_2}.$$

(b) Let $\tilde{\mathbf{x}} = \mathbf{V_1}\Sigma_1^{-1}\mathbf{U_1^T b}$ (*i.e.*, $\mathbf{z_2} = \mathbf{0}$), so $\tilde{\mathbf{z}} = \mathbf{V^T}\tilde{\mathbf{x}}$, implies $\tilde{\mathbf{z}} = \begin{bmatrix} \Sigma_1^{-1}\mathbf{c_1} \\ \mathbf{0} \end{bmatrix}$, then $\|\tilde{\mathbf{x}}\|_2 = \|\mathbf{V}\tilde{\mathbf{z}}\|_2 = \left\|\Sigma_1^{-1}\mathbf{c_1}\right\|_2$.

For any solution $\mathbf{x}$, we have

$$\|\mathbf{x}\|_2 = \|\mathbf{Vz}\|_2 = \left\|\begin{bmatrix} \Sigma_1^{-1}\mathbf{c_1} \\ \mathbf{z_2} \end{bmatrix}\right\|_2 \geq \left\|\Sigma_1^{-1}\mathbf{c_1}\right\|_2 = \|\tilde{\mathbf{x}}\|_2.$$

Finally,

$$\|\tilde{\mathbf{x}}\|_2 = \left\|\mathbf{V_1}\Sigma_1^{-1}\mathbf{U_1^T b}\right\|_2 = \left\|\Sigma_1^{-1}\mathbf{U_1^T b}\right\|_2 \leq \left\|\Sigma_1^{-1}\mathbf{U_1^T}\right\|_2 \|\mathbf{b}\|_2 = \left\|\Sigma_1^{-1}\right\|_2 \|\mathbf{b}\|_2 = \frac{\|\mathbf{b}\|_2}{\sigma}.$$

Problem 5. Consider the family of semi-implicit Runge-Kutta methods

$$k_1 = f(y_n + \beta h k_1), \quad k_2 = f(y_n + h k_1 + \beta h k_2),$$
$$y_{n+1} = y_n + h\Big((\tfrac{1}{2} + \beta)k_1 + (\tfrac{1}{2} - \beta)k_2\Big).$$

(a) Determine the order and the principal part of the local truncation error.

(b) Show that if $\beta > \frac{1}{2}$, then the negative real axis $\{z : \mathrm{Re}(z) < 0, \mathrm{Im}(z) = 0\}$ is contained in the region of absolute stability of the method.

Solution:

(a) Apply this method to the problem $f(y) = \lambda y$, we get

$$\begin{aligned}
k_1 = \lambda y_n + \beta\lambda h k_1 &\implies (1-\beta\lambda h)k_1 = \lambda y_n \\
&\implies k_1 = (1-\beta\lambda h)^{-1}\lambda y_n \\
k_2 = \lambda y_n + \lambda h k_1 + \beta\lambda h k_2 &\implies (1-\beta\lambda h)k_2 = \lambda y_n + (1-\beta\lambda h)^{-1}\lambda^2 h y_n \\
&\implies k_2 = (1-\beta\lambda h)^{-1}\lambda y_n + (1-\beta\lambda h)^{-2}\lambda^2 h y_n.
\end{aligned}$$

Then the method can be written as

$$\begin{aligned}
y_{n+1} &= y_n + (1-\beta\lambda h)^{-1}\lambda h y_n + (\tfrac{1}{2}-\beta)(1-\beta\lambda h)^{-2}\lambda^2 h^2 y_n \\
&= \Big(1 + (1-\beta z)^{-1} z + (\tfrac{1}{2}-\beta)(1-\beta z)^{-2} z^2\Big) y_n \quad (z := \lambda h) \\
&= \Big(1 + z\sum_{i=0}^{\infty}\beta^i z^i + (\tfrac{1}{2}-\beta) z^2 \Big(\textstyle\sum_{i=0}^{\infty}\beta^i z^i\Big)^2\Big) y_n \quad (|z\beta| < 1) \\
&= \Big(1 + z\Big(1 + \beta z + \beta^2 z^2 + \beta^3 z^3 + O(z^4)\Big) + (\tfrac{1}{2}-\beta) z^2 \Big(1 + \beta z + \beta^2 z^2 + O(z^3)\Big)^2\Big) y_n \\
&= \Big(1 + z + \tfrac{1}{2}z^2 + \left(\beta-\beta^2\right) z^3 + \left(\tfrac{3}{2}\beta^2 - 2\beta^3\right) z^4 + O(h^5)\Big) y_n.
\end{aligned}$$

Assume $y_n = y(x_n)$, the exact solution $y(x_{n+1}) = e^z y_n$ can be written as

$$y(x_{n+1}) = \Big(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{1}{24} + O(h^5)\Big) y_n.$$

Comparing the coefficients of z^3, we conclude that

- If $\beta - \beta^2 \neq \frac{1}{6}$, *i.e.* $\beta \neq \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$, then the method is second order, and $\tau_n \sim (\beta - \beta^2 - \frac{1}{6})h^3$.
- If $\beta - \beta^2 = \frac{1}{6}$, *i.e.* $\beta = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$, then $\frac{3}{2}\beta^2 - 2\beta^3 \neq \frac{1}{24}$, the method is third order, and $\tau_n \sim (\frac{3}{2}\beta^2 - 2\beta^3 - \frac{1}{24})h^4$.

(b) It suffices to show if $\beta > \frac{1}{2}$ and $z < 0$, then

$$-1 < 1 + \frac{z}{1-\beta z} + (\tfrac{1}{2}-\beta)\left(\tfrac{z}{1-\beta z}\right)^2 < 1.$$

First note $\beta > \frac{1}{2}$ and $z < 0$ imply $\frac{z}{1-\beta z} < 0$, and $(\frac{1}{2}-\beta)(\frac{z}{1-\beta z})^2 < 0$, hence

$$1 + \frac{z}{1-\beta z} + (\tfrac{1}{2}-\beta)\left(\tfrac{z}{1-\beta z}\right)^2 < 1.$$

Now it remains to show

$$-1 < 1 + \frac{z}{1-\beta z} + (\tfrac{1}{2}-\beta)\left(\tfrac{z}{1-\beta z}\right)^2.$$

Observing that $-\frac{1}{\beta} \leq \frac{z}{1-\beta z}$ (since $\beta z - 1 \leq \beta z$), we only need to verify

$$-1 < 1 - \frac{1}{\beta} + (\tfrac{1}{2}-\beta)\left(-\tfrac{1}{\beta}\right)^2.$$

For $\beta \neq 0$, this is equivalent to

$$0 < (2\beta - 1)^2,$$

which certainly holds for $\beta > \frac{1}{2}$.

Problem 6. Consider the Beam equation from mechanics with boundary conditions that model a cantilever beam:

$$\begin{aligned}
&u^{(4)} = f(x), \quad x \in (0,\ 1), \\
&u(0) = u'(0) = u''(1) = u'''(1) = 0.
\end{aligned} \tag{1}$$

(a) Recast this equation into a variational problem, stating the trial and test function spaces.

(b) Interpret the variational problem as an energy minimization problem, clearly stating the energy functional. Prove that the variational problem and the energy minimization problems are equivalent.

(c) Develop a CG(3) (cubic continuous Galerkin method) finite element method for this problem.

(d) Prove an *a priori* error estimate for this method in the energy norm:

$$\|e\|_E = \left(\int_0^1 (e'')^2 dx\right)^{\frac{1}{2}},$$

Where $e = u(x) - U(x)$, in which, $u(x)$ is the exact solution to VP (variational problem), $U(x)$ is the FEM (finite element method) solution.

(e) Prove an *a priori* error estimate for this method in the L_2 norm:

$$\|e\|_{L_2} =: \|e\| = \left(\int_0^1 e^2 dx\right)^{\frac{1}{2}}.$$

Solution:

(a) Multiply both sides of $u^{(4)} = f(x)$ with test function v and integrate on $[0,\ 1]$ to get

$$\int_0^1 u^{(4)} v dx = \int_0^1 f(x) v dx,$$

integration by parts twice yields

$$u''' v\Big|_0^1 - u'' v'\Big|_0^1 + \int_0^1 u'' v'' dx = \int_0^1 f(x) v dx.$$

Assume $v(0) = 0,\ v'(0) = 0$ so that $u''' v\Big|_0^1 - u'' v'\Big|_0^1 = 0$, then

$$\int_0^1 u'' v'' dx = \int_0^1 f(x) v dx.$$

Define

$$V = \{w : \int_0^1 w^2 + (w')^2 + (w'')^2 dx < \infty, \quad w(0) = w'(0) = 0\},$$

then the Variational Problem(VP) is:

Find $u \in V$, such that

$$\int_0^1 u'' v'' dx = \int_0^1 f(x) v dx, \quad \forall v \in V. \tag{2}$$

(b) Define the total energy $F : V \to \mathbb{R}$ as

$$F(w) = \tfrac{1}{2}\int_0^1 (w'')^2 dx - \int_0^1 f(x) w dx,$$

then the energy minimization problem(MP) is:

Find $u \in V$ such that

$$F(u) \le F(w), \quad \forall w \in V. \tag{3}$$

We can prove the equivalence of VP and MP:

(VP⟹MP) Assume $u \in V$ such that $\int_0^1 u''v''dx = \int_0^1 f(x)vdx$ for all $v \in V$. Let $w = u + v \in V$, then

$$\begin{aligned}
F(w) =& \tfrac{1}{2}\int_0^1 (u'' + v'')^2 dx - \int_0^1 f(x)(u+v)dx \\
=& \tfrac{1}{2}\int_0^1 (u'')^2 dx + \tfrac{1}{2}\int_0^1 (v'')^2 dx + \int_0^1 u''v''dx - \int_0^1 f(x)udx - \int_0^1 f(x)vdx \\
=& \left(\tfrac{1}{2}\int_0^1 (u'')^2 dx - \int_0^1 f(x)udx\right) + \left(\int_0^1 u''v''dx - \int_0^1 f(x)udx\right) + \tfrac{1}{2}\int_0^1 (v'')^2 dx \\
=& F(u) + 0 + \tfrac{1}{2}\int_0^1 (v'')^2 dx \\
\geq& F(u),
\end{aligned}$$

where the last equality is obtained by the definition of total energy and the fact that u is solution to the VP. Which implies solution to VP is also solution to MP.

(VP⟸ MP) Assume $u \in V$ such that $F(u) \leq F(w)$ for all $w \in V$. Let $g(\epsilon) = F(u + \epsilon v)$, here $v \in V$ is arbitrary but fixed, then $g'(0) = 0$.
Note that

$$g'(\epsilon) = \int_0^1 (u'' + \epsilon v'')v''dx - \int_0^1 fvdx,$$

substitute $\epsilon = 0$ and use the fact that $g'(0) = 0$, we have

$$\int_0^1 u''v''dx = \int_0^1 fvdx,\ \forall v \in V.$$

Which implies solution to MP is solution to VP.

(c) (i) Partition: Let τ_h: $0 = x_0 < \cdots < x_M < x_{M+1} = 1$ be a partition of $[0,\ 1]$, let $h_j = x_j - x_{j-1}$ for $j = 1, \ldots, M+1$ be the size of j-th mesh $I_j = [x_{j-1},\ x_j]$, define $h := \max_{1\leq j\leq M+1} h_j$.
(ii) Finite element space: Let $V_h^3 \subseteq V$ be our finite element space defined as

$$V_h^3 := \{u \in C^1(0,\ 1)| \quad u|_{I_j} \text{ is cubic polynomial for all } j = 1, \ldots, M+1, \text{ and } u(0) = u'(0) = 0\}. \tag{4}$$

(iii) CG(3) Finite Element Method: Find $U(x)$ in V_h^3 such that

$$\int_0^1 U''v''dx = \int_0^1 fvdx, \quad \forall v \in V_h^3.$$

(d) Let $u(x) \in V$ be the exact solution to VP (variational problem), $U(x) \in V_h^3$ be the FEM solution, we estiamte the error $e = u - U$ as follows:

$$\begin{aligned}
\|u\|_E^2 &= \int_0^1 (e'')^2\, dx \\
&= \int_0^1 e''(u - v + v - U)''dx \quad (v \in V_h^3) \\
&= \int_0^1 e''(u-v)''dx + \int_0^1 e''(v-U)''dx \quad \text{(by Galerkin orthogonality)} \\
&= \int_0^1 e''(u-v)''dx \quad \text{(by Cauchy inequality)} \\
&\leq \|u - U\|_E \cdot \|u - v\|_E\,.
\end{aligned}$$

Hence $\|e\|_E \leq \|u - v\|$ for all $v \in V_h^3$, take $v = \pi_h u$ as interpolation of u, then

$$\|e\|_E \leq Ch^2\|u^{(4)}\|,$$

where C comes from interpolation error.

(e) Consider the following dual problem

$$\phi^{(4)} = e, \quad \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0.$$

We have

$$\begin{aligned}
\|e\|^2 &= \int_0^1 e\phi^{(4)} dx \\
&= \int_0^1 e''\phi'' dx + e\phi'''\Big|_0^1 - e'\phi''\Big|_0^1 \quad \text{(by integration by parts twice)} \\
&= \int_0^1 e''\phi'' dx \quad \left(\text{subtract } \int_0^1 e''(\pi_h\phi)'' dx = 0\right) \\
&= \int_0^1 e''(\phi - \pi_h\phi)'' dx \\
&\le \|e\|_E \cdot \|\phi - \pi_h\phi\|_E \quad \text{(by the interpolation error and energy norm estimate)} \\
&\le C^2 h^4 \|u^{(4)}\| \cdot \|\phi^{(4)}\| \quad \left(\text{ since } \|\phi^{(4)}\| = \|e\|\right) \\
&= C^2 h^4 \|u^{(4)}\| \cdot \|e\|,
\end{aligned}$$

that is, $\|e\| \le C^2 h^4 \|u^{(4)}\|$.