S.-T. Yau College Student Mathematics Contests 2017

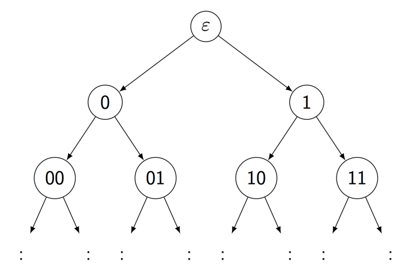
Probability and Statistics

Individual (5 problems)

Problem 1. A box contains 750 red balls and 250 blue balls. Repeatedly pick a ball uniformly at random from the box and remove it until all remaining balls have a single color. (Note: no replacement).

Please find integer m such that the expectation value for the total number of the remaining balls $\in [m, m+1]$

Problem 2. Suppose a number $X_0 \in \{1, -1\}$ at the root of a binary tree



is propagated away from the root as follows. The root is the node at level 0. After obtaining the 2^h numbers at the nodes at level h, each number at level $h+1$ is obtained from the number adjacent to it (at level h) by flipping its sign with probability $p \in (0, 1/2)$ independently.

Let X_h be the average of the 2^h values received at the nodes at level h. Define the *signal-to-noise ratio* at level h to be

$$R_h := \frac{\big(\mathbb{E}[X_h \,|\, X_0 = 1] - \mathbb{E}[X_h \,|\, X_0 = -1]\big)^2}{Var[X_h | X_0 = 1]}.$$

Find the threshold number p_c such that R_h converges to 0 if $p \in (p_c\,,\, 1/2)$ and diverges if $p \in (0,\, p_c)$, as $h \to \infty$.

Problem 3. Consider the space representing an infinite sequence of coin flips, namely $\Omega := \{H, T\}^\infty$, (H: head, T: tail) with the associated σ-field $\mathcal{F}$ generated by finite dimensional rectangles. For $0 \le p \le 1$, denote by $\mathbb{P}_p$ the probability measure on $(\Omega, \mathcal{F})$ corresponding to flipping a coin an infinite number of times with probability of H being p and probability of T being $q = 1 - p$ at each flip.

Show that for each $p \in [0, 1]$, there exists A_p such that

$$\mathbb{P}_p(A_p) > 1/2$$

and for any $p' \neq p$, $p' \in [0,1]$

$$\mathbb{P}_{p'}(A_p) < 1/2$$

Problem 4. Let $G := G(n,p)$ be a random graph with n vertices where each possible edge has probability p of existing. The existence of the edges are independent to each other. With G, we say $A \subset \{1, 2, \cdots, n\}$ is a fully connected set if and only if

$$i, j \in A \implies i-th \text{ and } j-th \text{ vertices are (directly) connected with an edge in } G$$

Define T as the size of the largest fully connected set

$$T := \max\{|A| : \text{ A is a fully connected set}\}$$

Let's fix $p \in (0,1)$, please prove that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T}{2\log_{\frac{1}{p}} n} \leq 1+\epsilon\right) = 1, \quad \forall \epsilon > 0,$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T}{\sqrt{2\log_{\frac{1}{p}} n}} \geq 1-\epsilon\right) = 1, \quad \forall \epsilon > 0,$$

Hint:

$$\mathbb{P}\left(T = n\right) = p^{\binom{n}{2}} = p^{n(n-1)/2}$$

Problem 5. Consider a population of constant size $N+1$ that is suffering from an infectious disease. We can model that spread of the disease as Markov process. Let $X(t)$ be the number of healthy individuals at time t and suppose that $X(0) = N$. We assume that if $X(t)$

$$\lim_{h\to 0} \frac{1}{h} \mathbb{P}\left(X(t+h) = n-1 | X(t) = n)\right) = \lambda n \left(N+1-n\right)$$

For $0 \leq s \leq 1$, $0 \leq t$, define

$$G(s,t) := \mathbb{E}(s^{X(t)})$$

Please find a non-trivial partial differential equation for $G(s,t)$, which involves $\partial_t G$.