S.-T. Yau College Student Mathematics Contests 2011

Geometry and Topology

Individual
9:30–12:00 am, July 10, 2011
(Please select 5 problems to solve)

1. Suppose M is a closed smooth n-manifold.
a) Does there always exist a smooth map $f : M \to S^n$ from M into the n-sphere, such that f is essential (i.e. f is not homotopic to a constant map)? Justify your answer.
b) Same question, replacing S^n by the n-torus T^n.

2. Suppose (X, d) is a compact metric space and $f : X \to X$ is a map so that $d(f(x), f(y)) = d(x, y)$ for all x, y in X. Show that f is an onto map.

3. Let C_1, C_2 be two linked circles in $\mathbb{R}^3$. Show that C_1 cannot be homotopic to a point in $\mathbb{R}^3 \backslash C_2$.

4. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.

5. A regular curve C in $\mathbb{R}^3$ is called a *Bertrand Curve,* if there exists a diffeomorphism $f : C \to D$ from C onto a different regular curve D in $\mathbb{R}^3$ such that $N_xC = N_{f(x)}D$ for any $x \in C$. Here N_xC denotes the principal normal line of the curve C passing through x, and T_xC will denote the tangent line of C at x. Prove that:
a) The distance $|x - f(x)|$ is constant for $x \in C$; and the angle made between the directions of the two tangent lines T_xC and $T_{f(x)}D$ is also constant.
b) If the curvature k and torsion τ of C are nowhere zero, then there must be constants λ and μ such that $\lambda k + \mu\tau = 1$

6. Let M be the closed surface generated by carrying a small circle with radius r around a closed curve C embedded in $\mathbb{R}^3$ such that the center moves along C and the circle is in the normal plane to C at each point. Prove that

$$\int_M H^2 d\sigma \geq 2\pi^2,$$

and the equality holds if and only if C is a circle with radius $\sqrt{2}r$. Here H is the mean curvature of M and $d\sigma$ is the area element of M.