S.-T. Yau College Student Mathematics Contests 2021

Geometry and Topology

Solve every problem.

Problem 1.

(a) Show that $\mathbf{P^{2n}}$ can not be the boundary of a compact manifold.

(b) Show that $\mathbf{P^3}$ is the boundary of some compact manifold.

Problem 2. Suppose M is a noncompact, complete n-dimensional manifold, and suppose there is an open subset $U \subset M$ and an open set $V \subset \mathbf{R^n}$ such that $M \backslash U$ is isomorphic to $\mathbf{R^n} \backslash V$. If $\mathrm{Ric} M \geq 0$, show that M is isometric to $\mathbf{R^n}$.

Problem 3. Compute all the homotopy groups of the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$, $n \geq 2$.

Problem 4. Consider the upper half space $\mathbf{H^3} = \{(x, y, z) \mid z > 0\}$ equipped with hyperbolic metric $g = \dfrac{dx^2 + dy^2 + dz^2}{z^2}$. Let P be the surface defined by $\{z = x \tan \alpha,\ z > 0\}$ for some $\alpha \in (0, \frac{\pi}{2})$. Compute the mean curvature of P.

Problem 5. Suppose M is a compact 2-dimensional Riemannian manifold without boundary, with positive sectional curvature. Show that any two compact closed geodesics on M must intersect with each other.

Problem 6. Suppose Σ is a smooth compact embedded hypersurface (*i.e.* a codimension 1 submanifold) without boundary in $\mathbf{R^n}$ for $n \geq 3$. Show that Σ is orientable.