S.-T. Yau College Student Mathematics Contests 2017

Algebra and Number Theory

Individual

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.** Let $\mathbb{Q}_p$ denote the field of p-adic numbers and let $\mathbb{Z}_p$ denote the ring of p-adic integers (p is a prime number).

(a) (5 points) Show that for every integer $k \geq 0$, $(p^{-k}\mathbb{Z}_p)/\mathbb{Z}_p \cong \mathbb{Z}/p^k\mathbb{Z}$ as abelian groups.

(b) (5 points) Determine the endomorphism ring of the abelian group $(p^{-k}\mathbb{Z}_p)/\mathbb{Z}_p$ ($k \geq 0$).

(c) (5 points) Determine the endomorphism ring of the abelian group $\mathbb{Q}_p/\mathbb{Z}_p$.

(d) (5 points) Determine the endomorphism ring of the abelian group $\mathbb{Q}/\mathbb{Z}$.

Problem 2 (20 points)**.** Let A be a finite abelian group and let $\phi : A \to A$ be an endomorphism. Put

$$A_{\mathrm{nil}} := \{x \in A \mid \phi^k(x) = 0 \text{ for some } k \geq 1\}.$$

(a) (15 points) Show that there is a subgroup A_0 of A such that ϕ restricts to an automorphism of A_0 and $A = A_0 \oplus A_{\mathrm{nil}}$.

(b) (5 points) Show that such a subgroup is unique.

Problem 3 (20 points)**.** Let L/F be a Galois field extension, not necessarily finite. Let $x \in L$.

(a) (6 points) Show that the set $\mathcal{P}$ of subextensions of L/F not containing x has a maximal element E. Let K/E be a nontrivial finite extension contained in L. Show that $x \in K$.

(b) (6 points) Let K' be the Galois closure of K/E in L. Show that there exists $g \in G = \mathrm{Gal}(K'/E)$ such that $gx \neq x$.

(c) (8 points) Deduce that K/E is a cyclic Galois extension.

Problem 4 (20 points)**.** The goal of this problem is to prove the Chevalley–Warning theorem. Let p be a prime number and q a power of p.

(a) (8 points) Let $0 \leq a < q-1$ be an integer. Show that $S(X^a) := \sum_{x \in \mathbb{F}_q} x^a$ equals 0. Here we adopt the convention $x^0 = 1$ in $\mathbb{F}_q$ even for $x = 0$.

(b) (12 points) Let $f_1, \ldots, f_m \in \mathbb{F}_q[X_1, \ldots, X_n]$ be polynomials in n variables satisfying

$$\sum_{i=1}^{m} \deg(f_i) < n.$$

Show that $P = \prod_{i=1}^{m}(1 - f_i^{q-1})$ satisfies

$$S(P) := \sum_{(x_1,\ldots,x_n) \in \mathbb{F}_q^n} P(x_1, \ldots, x_n) = 0.$$

Deduce that p divides the cardinality of the set

$$V = \{(x_1, \ldots, x_n) \in \mathbb{F}_q^n \mid f_i(x_1, \ldots, x_n) = 0 \quad \forall i\}.$$

Problem 5 (20 points)**.** In this problem, all matrices are $n \times n$ with complex entries. Let U and V be matrices such that $UV \neq VU$. Assume that U is diagonalizable and commutes with VUV^{-1}.

(a) (10 points) For $\lambda, \mu \in \mathbb{C}$, let

$$E_{\lambda,\mu} = \{x \in \mathbb{C}^n \mid Ux = \lambda x, \quad VUV^{-1}x = \mu x\}.$$

Show that there exist couples $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$, satisfying $\lambda_i \neq \mu_i$ and $E_{\lambda_i,\mu_i} \neq 0$ for $i = 1, 2$.

(b) (10 points) For a matrix A, we define $\mathrm{N}(A) := \mathrm{tr}(A^*A)$, where $A^* = \bar{A}^T$ is the conjugate transpose of A. Assume that U and V are unitary (namely, $U^*U = V^*V$ is the identity matrix). Deduce that $\mathrm{N}(1+V) \geq 4$.