S.-T. Yau College Student Mathematics Contests 2010

Geometry and Topology

Individual

(Please select 5 problems to solve)

1. Let $D^* = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ be the punctured unit disc in the Euclidean plane. Let g be the complete Riemannian metric on D^* with contsant curvature -1. Find the disctance under the metric between the points $(e^{-2\pi}, 0)$ and $(-e^{-\pi}, 0)$.

2. Show that every closed hypersurface in $\mathbb{R}^n$ has a point at which the second fundamental form is positive definite.

3. Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd.

4. Suppose $\pi : M_1 \longrightarrow M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_pM_1 \longrightarrow T_{\pi(p)}M_2$ is a vector space isomorphism.
(a). Show that if M_1 is connected, then π is a covering space projection.
(b). Given an example where M_2 is compact but $\pi : M_1 \longrightarrow M_2$ is not a covering space (but has the π_* isomorphism property).

5. Let Σ_g be the closed orientable surface of genus g. Show that if $g > 1$, then Σ_g is a covering space of Σ_2.

6. Let M be a smooth 4-dimensional manifold. A symplectic form is a closed 2-form ω on M such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.
(a). Construct a symplectic form on $\mathbb{R}^4$.
(b). Show that there are no symplectic forms on S^4.