S.-T. Yau College Student Mathematics Contests 2023

Geometry and Topology

Solve every problem.

1. (a) Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra. The *Maurer-Cartan* form on G is the unique left-invariant $\mathfrak{g}$-valued 1-form such that $\omega|_e : T_eG \to \mathfrak{g}$ is the identity map, where e is the identity in G. Show that the *Maurer-Cartan form* ω satisfies the *Maurer-Cartan equation* $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

   (b) Let G be a matrix group $GL(n, \mathbb{R})$, give detailed computation to find the *Maurer-Cartan form* ω.

   (c) Let $SE(3)$ be the special Euclidean group, i.e. it contains all transformations of $\mathbb{R}^3$ (as 3-dim Euclidean space) of the form $\mathbf{x} \mapsto t + A\mathbf{x}$, where $\mathbf{t} \in \mathbb{R}^{\mathbf{3}}$ and $A \in SO(3)$. Find an expression of the Maurer Cartan form ω of $SE(3)$, also check it satisfies the standard Cartan structure equations in Eulidean space.

2. Let M, N be closed, connected, oriented 3-manifolds with the first fundamental groups

$$\pi_1(M_1) = \mathbb{Z}_3 \oplus \mathbb{Z}^2, \quad \pi_1(M_2) = \mathbb{Z}_6 \oplus \mathbb{Z}^3,$$

   (a) Find all homology groups $H_n(M_1, \mathbb{Z})$ and $H_n(M_2, \mathbb{Z})$.

   (b) Find all homology groups $H_n(M_1 \times M_2, \mathbb{Q})$.

   (c) Does there exist a closed connected oriented 3-manifold M with

$$\pi_1(M) = \mathbb{Z}_3 \oplus \mathbb{Z}^2 \quad \text{or} \quad \pi_1(M) = \mathbb{Z}_6 \oplus \mathbb{Z}^3?$$

3. (a) Let f be a diffeomorphism goup of a circle S^1, assume f has no fixed point and it is generated by a smooth vector field, show that f must be conjugate to a rotation.

   (b) Show that there is a diffeomorphism $f : S^1 \to S^1$, such that f can not be generated by a smooth vector field but it is arbitrarily closed the identity map $i : S^1 \to S^1$ in C^∞-topology.

4. (a) State the Leray-Hirsh theorem.

   (b) Let

$$\begin{aligned} Fl_k(\mathbb{C}^n) &= \{ \text{ all k-flags in } \mathbb{C}^n\} \\ &= \{(F_0, \cdots, F_k) |\ F_i \text{ is an } i\text{-dim subspace of } \mathbb{C}^n, \text{ s.t. } F_0 \subset F_1 \subset \cdots \subset F_k \subset \mathbb{C}^n\} \end{aligned}$$

   Let $\Phi : Fl_k(\mathbb{C}^n) \to Fl_{k-1}(\mathbb{C}^n)$ be the projection map sending a k-flag $(F_0, \cdots, F_k)$ to a $(k-1)$-flag $(F_0, \cdots, F_{k-1})$, it is known this is a fiber bundle. What is the fiber of Φ?

   (c) Compute the Euler Characteristic $\chi(Fl_n(\mathbb{C}^n))$.

5. On the Euclidean space $\mathbb{R}^n$, we consider an $n-1$ form α, which is of class C^1, such that both α and $d\alpha$ are in L^1. Show that $\int_{\mathbb{R}^n} d\alpha = 0$.

6. A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a gradient expanding Ricci soliton if there exists a smooth function f on M^n such that the Ricci tensor Ric of the metric g is given by

$$Ric + \text{Hess} f = \lambda g,$$

   for some negative constant $\lambda < 0$. Show that if M is compact, then a gradient expanding Ricci soliton must be an Einstein metric.