S.-T. Yau College Student Mathematics Contests 2015

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Let $\phi \in C([a,b],R)$. Suppose for every function $h \in C^1([a,b],R), h(a) = h(b) = 0$, we have

$$\int_a^b \phi(x)h(x)dx = 0.$$

Prove that $\phi(x) = 0$.

2. Let f be a Lebesgue integrable function over $[a, b+\delta], \delta > 0$, prove that

$$\lim_{h\to 0+} \int_a^b |f(x+h) - f(x)|dx \to 0.$$

3. Let $L(q, q^{'}, t)$ be a function of $(q, q^{'}, t) \in TU \times R, U$ is an open domain in R^n. Let $\gamma : [a,b] \to U$ be a curve in U. Define a functional $S(\gamma) = \int_a^b L(\gamma(t), \gamma^{'}(t), t)dt$. We say that γ is an extremal if for every smooth variation of $\gamma, \phi(t,s), s \in (-\delta, \delta), \phi(t,0) = \gamma(t), \phi_s = \phi(t,s)$, we have $\frac{dS(\phi_s)}{ds}|_{s=0} = 0$. Prove that every extremal γ satisfies the Euler-Lagrange equation: $\frac{d}{dt}(\frac{\partial L}{\partial q^{'}}) = \frac{\partial L}{\partial q}$.

4. Let $f : U \to U$ be a holomorphic function with U a bounded domain in the complex plane. Assuming $0 \in U, f(0) = 0, f^{'}(0) = 1$, prove that $f(z) = z$.

5. Let $T : H_1 \to H_2$ be a bounded operator of Hilbert spaces H_1, H_2. Let $S : H_1 \to H_2$ be a compact operator, that is, for every bounded sequence $\{v_n\} \in H_1, Sv_n$ has a converging subsequence. Show that $Coker(T+S) = H_2/\overline{Im(T+S)}$ is finite dimensional and $Im(T+S)$ is closed in H_2. (Hint: Consider equivalent statements in terms of adjoint operators.)

6. Let $u \in C^2(\bar{\Omega}), \Omega \subset R^d$ is a bounded domain with a smooth boundary.

1) Let u be a solution of the equation $\Delta u = f, u|_{\partial\Omega} = 0, f \in L^2(\Omega)$. Prove that there is a constant C depends only Ω such that

$$\int_\Omega (\Sigma_{j=1}^n (\frac{\partial u}{\partial x_j})^2 + u^2)dx \leq C \int_\Omega f^2(x)dx.$$

2) Let $\{u_n\}$ be a sequence of harmonic functions on Ω, such that $||u_n||_{L^2(\Omega)} \leq M < \infty$, for a constant M independent of n. Prove that there is a converging subsequence $\{u_{n_k}\}$ in $L^2(\Omega)$.