GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Geometry and Topology

Please solve 5 out of the following 6 problems.

1. Prove that the real projective space $\mathbb{RP}^n$ is a differentiable manifold of dimension n.

2. Let M, N be n-dimensional smooth, compact, connected manifolds, and $f : M \to N$ a smooth map with rank equals to n everywhere. Show that f is a covering map.

3. Given any Riemannian manifold (M^n, g), show that there exists a unique Riemannian connection on M^n.

4. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and $f : S^n \to S^n$ a continuous map. Assume that the degree of f is an odd integer. Show that there exists $x_0 \in S^n$ such that $f(-x_0) = -f(x_0)$.

5. State and prove the Stokes theorem for oriented compact manifolds.

6. Let M be a surface in $\mathbb{R}^3$. Let D be a simply-connected domain in M such that the boundary ∂D is compact and consists of a finite number of smooth curves. Prove the Gauss-Bonnet Formula:

$$\int_{\partial D} k_g \, ds \; + \sum_j (\pi - \alpha_j) + \iint_D K \, dA \; = \; 2\pi,$$

where k_g is the geodesic curvature of the boundary curve. Each α_j is the interior angle at a vertex of the boundary, K is the Gaussian curvature of M, and the 2-form dA is the area element of M.