S.-T. Yau College Student Mathematics Contests 2019

Geometry and Topology

Individual (5 problems)

1) Let Conf_n be the following submanifold of $\mathbb{C}^n$:

$$\mathrm{Conf}_n = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for any } i \neq j\}.$$

For every pair (i, j) with $i \neq j$, we define the complex valued 1-form

$$\omega_{ij} := \frac{dz_i - dz_j}{z_i - z_j}.$$

(a) Show that for any $i \neq j$, ω_{ij} represents a non-zero de Rham cohomology class in $H^1(\mathrm{Conf}_n, \mathbb{C})$.

(b) Show that for any pair-wise distinct indices i, j, k,

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0.$$

2) Let M be a compact oriented manifold of (real) dimension 4. Consider the following symmetric bilinear form on $H^2(M)$

$$H^2(M) \times H^2(M) \to \mathbb{R}, \quad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta.$$

Let $\tau(M)$ be the signature of this bilinear form, i.e. the number of positive eigenvalues minus the number of negative eigenvalues. Compute $\tau(M)$ for $M = S^4, \mathbb{CP}^2$ and $S^2 \times S^2$.

3) Let $X = \mathbb{R}^4 / \sim$, where

$$\begin{aligned}
(x_1, x_2, x_3, x_4) &\sim (x_1, x_2 + 1, x_3, x_4) \\
(x_1, x_2, x_3, x_4) &\sim (x_1, x_2, x_3, x_4 + 1) \\
(x_1, x_2, x_3, x_4) &\sim (x_1 + 1, x_2, x_3, x_4) \\
(x_1, x_2, x_3, x_4) &\sim (x_1, x_2 + x_4, x_3 + 1, x_4)
\end{aligned}$$

Compute $H_1(X, \mathbb{Z})$.

4) Let E be a vector bundle over a smooth manifold M. Let ∇^E be a connection E and $R^E \in \Omega^2(M, \mathrm{End}(E))$ be its curvature tensor. For any polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, we denote

$$f(R^E) = a_0 + a_1 R^E + a_2 (R^E)^2 \cdots + a_n (R^E)^n \in \Omega^*(M, \mathrm{End}(E)).$$

Here $(R^E)^k \in \Omega^{2k}(M, \mathrm{End}(E))$ is the k-th wedge product on forms combined with matrix multiplications on $\mathrm{End}(E)$.

(a) Show that the differential form $\mathrm{tr}\left[f(R^E)\right] \in \Omega^*(M)$ is closed

$$d\mathrm{tr}\left[f(R^E)\right] = 0.$$

Here tr is the trace on $\mathrm{End}(E)$.

(b) Let ∇^E, $\widetilde{\nabla}^E$ be two connections on E and R^E, $\widetilde{R}^E$ be the corresponding curvature tensors. Show that there exists a differential form $\omega \in \Omega^*(M)$ such that

$$\mathrm{tr}\left[f(R^E)\right] - \mathrm{tr}\left[f(\tilde{R}^E)\right] = d\omega.$$

5) (a) Let u be a smooth function over a Riemannian manifold (M, g). Prove the following Bochner's formula

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla\nabla u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + g(\nabla\Delta u, \nabla u)$$

where Δ is the Laplacian and $|\bullet|^2 = g(\bullet, \bullet)$.

(b) Let (S^2, g) be the standard unit sphere and E be a constant. Show that the only smooth positive solution to

$$\Delta \ln f + Ef^2 = 1$$

is $f = \frac{1}{A+\phi}$ where A is a constant and ϕ is some first eigenfunction of S^2.