S.-T. Yau College Student Mathematics Contests 2020

Geometry and Topology

Solve every problem.

Problem 1. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$.

(a) Find a 6-form α on $\mathbb{R}^7 \setminus \{0\}$ such that

$$d\alpha = 0, \quad \text{and} \quad \int_{S^6} \alpha = 1.$$

(b) For any smooth map $f : S^{11} \to S^6$, show that there exists a 5-form φ on S^{11} such that

$$f^*\alpha = d\varphi. \tag{1}$$

(c) Let

$$H(f) = \int_{S^{11}} \varphi \wedge d\varphi.$$

Show that $H(f)$ is independent of the choice of φ satisfying (1).

(d) Show that $H(f)$ is an even integer, for any smooth map $f : S^{11} \to S^6$.

Problem 2. For any $h \in C^\infty(\mathbb{R}^2)$ and $h > 0$ on $\mathbb{R}^2$, define the Ricci curvature $\mathrm{Ric}(h)$ associated with h by

$$\mathrm{Ric}(h) = \frac{1}{h}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) \log h,$$

where (x, y) are the standard Cartesian coordinates in $\mathbb{R}^2$. Either construct a positive smooth function h_1 such that $\mathrm{Ric}(h_1) = 1$, or show that no such function h_1 exists.

Problem 3. Let M be an n-dimensional Riemannian manifold, and $p \in M$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space T_pM, and let $\{x^1, \dots, x^n\}$ be a coordinate system of M centered at p such that

$$\exp_p^{-1}(q) = \sum_{j=1}^{n} x^j(q) e_j,$$

where $\exp_p$ denotes the exponential map. Let $\gamma(t) = \exp_p(te_1)$, $0 \le t \le \delta$, where δ is a positive constant less than 1.

(a) For $2 \le \alpha \le n$, which one of the following,

$$t\frac{\partial}{\partial x^\alpha}\Big|_{\gamma(t)} \quad \text{or} \quad \frac{\partial}{\partial x^\alpha}\Big|_{\gamma(t)},$$

is a Jacobi field along $\gamma(t)$? Prove your assertion.

(b) Denote

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, \quad 1 \le i, j \le n.$$

Compute

$$\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} \quad \text{at the point } p.$$

(c) Show that

$$\max_{0 \le t \le \delta} \left| \frac{\partial g_{22}}{\partial x^1}(\gamma(t)) \right| \le C\delta A,$$

where $C > 0$ is a constant depending only on n, and A is the C^0-bound of the curvature tensor of M along $\gamma(t)$, for $0 \le t \le \delta$.

Problem 4. Let $\mathrm{SO}(n)$ be the set of $n \times n$ orthogonal real matrices with determinant equal to 1. Endow $\mathrm{SO}(n)$ with the relative topology as a subspace of Euclidean space $\mathbb{R}^{n^2}$.

(a) Show that $\mathrm{SO}(n)$ is compact.

(b) Is $\mathrm{SO}(3)$ homeomorphic to the real projective space $\mathbb{RP}^3$? Prove your assertion.

(c) Compute the fundamental group of $\mathrm{SO}(2020)$.

Problem 5. Let X be a topological space and $\pi : \mathbb{R}^2 \to X$ a covering map. Let K be a compact subset of X and B the closed unit ball centered at the origin in $\mathbb{R}^2$.

(a) Suppose $\pi : \mathbb{R}^2 \setminus B \to X \setminus K$ is a homeomorphism. Show that $\pi : \mathbb{R}^2 \to X$ is a homeomorphism.

(b) Suppose $\mathbb{R}^2 \setminus B$ is homeomorphic to $X \setminus K$, where the homeomorphism may not be π. Is X necessarily homeomorphic to $\mathbb{R}^2$? Prove your assertion.

Problem 6. Let F_n be the free group of rank n,

(a) Give an example of a finite connected graph such that its fundamental group is F_2.

(b) Does F_2 contain a proper subgroup isomorphic to F_2?

(c) Does F_2 contain a proper finite index subgroup isomorphic to F_2?