S.-T. Yau College Student Mathematics Contests 2014

Applied Math. and Computational Math.

Team

Please solve as many problems as you can!

1. (15 pts)
Given a finite positive (Borel) measure $d\mu$ on $[0,1]$, define its sequence of moments as follows

$$c_j = \int_0^1 x^j \, d\mu(x)\,, \quad j = 0, 1, \ldots .$$

Show that the sequence is *completely monotone* in the sense that that

$$(I - S)^k c_j \geq 0 \quad \text{for all } j, k \geq 0,$$

where S denotes the backshift operator given by $Sc_j = c_{j+1}$ for $j \geq 0$.

2. (20 pts)
We recall that a polynomial

$$f(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$

is called an Eisenstein polynomial if for some prime p we have

(i) $p \mid a_i$ for $i = 0, \ldots, d-1$,
(ii) $p^2 \nmid a_0$,
(iii) $p \nmid a_d$.

Eisenstein polynomials are well-know to be irreducible over $\mathbb{Z}$, so they can be used to construct explicit examples of irreducible polynomials.

Questions:

(i) Prove that a composition $f(g(X))$ of two Eisenstein polynomials f and g is an Eisenstein polynomial again.
(ii) Suggest a multivariate generalisation of the Eisenstein polynomials. That is, describe a class polynomials $F(X_1, \ldots, X_m)$ in terms of the divisibility properties of their coefficients that are guaranteed to be irreducible.

3. (20 pts) For solving the following partial differential equation

$$u_t + f(u)_x = 0, \qquad 0 \leq x \leq 1 \tag{1}$$

where $f'(u) \geq 0$, with periodic boundary condition, we can use the following semi-discrete discontinuous Galerkin method: Find $u_h(\cdot, t) \in V_h$ such that, for all $v \in V_h$ and $j = 1, 2, \cdots, N$,

$$\int_{I_j} (u_h)_t v dx - \int_{I_j} f(u_h) v_x dx + f((u_h)^-_{j+1/2}) v^-_{j+1/2} - f((u_h)^-_{j-1/2}) v^+_{j-1/2} = 0, \tag{2}$$

with periodic boundary condition

$$(u_h)^-_{1/2} = (u_h)^-_{N+1/2}; \quad (u_h)^+_{N+1/2} = (u_h)^+_{1/2}, \tag{3}$$

where $I_j = (x_{j-1/2}, x_{j+1/2})$, $0 = x_{1/2} < x_{3/2} < \cdots < x_{N+1/2} = 1$, $h = \max_j(x_{j+1/2} - x_{j-1/2})$, $v^\pm_{j+1/2} = v(x^\pm_{j+1/2}, t)$, and

$$V_h = \{v : v|_{I_j} \text{ is a polynomial of degree at most } k \text{ for } 1 \le j \le N\}.$$

Prove the following L^2 stability of the scheme

$$\frac{d}{dt}E(t) \le 0 \tag{4}$$

where $E(t) = \int_0^1 (u_h(x,t))^2 dx$.

4. Consider the linear system $Ax = b$. The GMRES method is a projection method which obtains a solution in the m-th Krylov subspace K_m so that the residual is orthogonal to AK_m. Let r_0 be the initial residual and let $v_0 = r_0$. The Arnoldi process is applied to build an orthonormal system $v_1, v_2, \cdots, v_{m-1}$ with $v_1 = Av_0/\|Av_0\|$. The approximate solution is obtained from the following space

$$K_m = \text{span}\{v_0, v_1, \cdots, v_{m-1}\}.$$

(i) (5 points) Show that the approximate solution is obtained as the solution of a least-square problem, and that this problem is triangular.

(ii) (5 points) Prove that the residual r_k is orthogonal to $\{v_1, v_2, \cdots, v_{k-1}\}$.

(iii) (5 points) Find a formula for the residual norm.

(iv) (5 points) Derive the complete algorithm.

5. (10 pts)

(i) Set $x_0 = 0$. Write the recurrence

$$x_k = 2x_{k-1} + b_k, \quad k = 1, 2, \cdots, n,$$

in a matrix form $A\vec{x} = \vec{b}$. For $b_1 = -1/3$, $b_k = (-1)^k, k = 2, 3, \cdots, n$, verify that $x_k = (-1)^k/3, k = 1, 2, \cdots, n$ is the exact solution.

(ii) Find A^{-1} and compute condition number of A in L^1 norm.