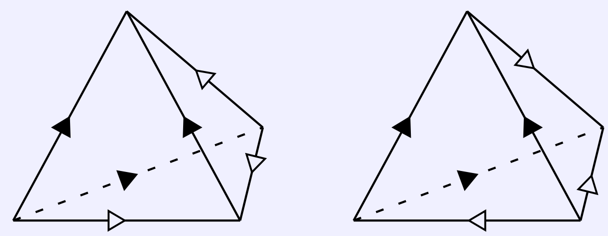
S.-T. Yau College Student Mathematics Contests 2022

Geometry and Topology

Solve every problem.

Problem 1. The topological space X is obtained by gluing two tetrahedra as illustrated by the figure. There is a unique way to glue the faces of one tetrahedron to the other so that the arrows are matched. The resulting complex has 2 tetrahedra, 4 triangles, 2 edges and 1 vertex.

Show that X can not have the homotopy type of a compact manifold without boundary.



Solution: One can calculate the ($\mathbf{Z}$-coefficient) simplicial homology to see that $H_0(X) = \mathbf{Z}$, $H_1(X) = \mathbf{Z}^2$, $H_2(X) = \mathbf{Z}/2$, $H_3(X) = \mathbf{Z}$. This does not satisfy Poincaré duality, hence the X can not have the homotopy type of a compact manifold without boundary. Or one can notice that X has Euler characteristic 1, but a closed odd-dimensional manifold has Euler characteristic 0.

Problem 2. Suppose (M, h) is a closed (*i.e.*, compact without boundary) Riemannian manifold, and h is a metric on M with $\sec(h) \leq -1$, where $\sec(h)$ is the sectional curvature. Suppose Σ is a closed minimal surface with genus g in (M, h). Show that

$$\mathrm{Area}(\Sigma) \leq 4\pi(g-1).$$

Remark: *A minimal surface is an immersed surface with constant mean curvature* 0.

Solution: For any surface Σ in a Riemannian manifold (M, h), let $x \in \Sigma$, and $\{e_1, e_2, e_3, e_4, \ldots, e_n\}$ be a local orthonormal frame of M at x where e_1 and e_2 are tangent to Σ and $e_3, \ldots, e_n$ are normal to Σ. The Gauss equation shows that

$$\begin{aligned}\varkappa(x) =& K_{12} + \langle A_{11}, A_{22}\rangle - \langle A_{12}, A_{12}\rangle, \\ =& K_{12} + \frac{|H|^2}{2} - \frac{|A|^2}{2}.\end{aligned}$$

Here, K_{12} is the sectional curvature of $T_x\Sigma \subset TM$. Integrate this identity over Σ and use the Gauss-Bonnet theorem; we get

$$2\pi\chi(\Sigma) = \int_\Sigma \left(K_{12} + \frac{|H|^2}{2} - \frac{|A|^2}{2}\right).$$

Equivalently,

$$\mathrm{Area}(\Sigma) = 4\pi(g-1) + \int_\Gamma \left(1 + K_{12} + \frac{|H|^2}{2} - \frac{|A|^2}{2}\right),$$

where g is the genus of Σ. Σ being minimal implies that $H = 0$, and $\sec(h) \leq -1$ implies that $K_{12} \leq -1$. Se we get

$$\mathrm{Area}(\Sigma) \leq 4\pi(g-1).$$

Problem 3. For any topological space X, the *n-th symmetric product* of X is the quotient of the Cartesian product $(X)^n$ by the action of the symmetric group S_n, which permutes the factors in $(X)^n$. This space is denoted by $\mathrm{SP}^n(X)$, and the topology is the natural quotient topology induced from $(X)^n$.

Show that $\mathrm{SP}^n(\mathbf{CP}^1)$ is homeomorphic to $\mathbf{CP}^n$. Here $\mathbf{CP}^1$ and $\mathbf{CP}^n$ are equipped with the manifold topology.

Solution: $\mathbf{CP}^n$ can be interpreted as the space of homogeneous polynomials in two variables of degree n modulo multiplication by a non-zero complex constant. Each polynomial is determined up to a constant complex number by its n complex roots on $\mathbf{CP}^1$. On the other hand, $\mathrm{SP}^n(\mathbf{CP}^1)$ is exactly the n-tuples of unordered points in $\mathbf{CP}^1$. This induces a bijection $F : \mathrm{SP}^n(\mathbf{CP}^1) \to \mathbf{CP}^n$.

It remains to show F and F^{-1} are both continuous. One direction is relatively easy: because the coefficients of the polynomials are determined by the roots via Vieta's formulas, and Vieta's formulas are polynomials, F is continuous. For the other direction, notice that $\mathrm{SP}^n(\mathbf{CP}^1)$ is compact (because it is the quotient of a compact space), $\mathbf{CP}^n$ is Hausdorff, and F is a continuous bijection, so F^{-1} is also a continuous bijection.

Problem 4. Let M be a complete noncompact Riemannian manifold. M is said to have the *geodesic loops to infinity property* if for any $[\alpha] \in \pi_1(M)$ and any compact subset $K \subset M$, there is a geodesic loop $\beta \subset M \backslash K$, such that β is homotopic to α.

Show that if a complete noncompact Riemannian manifold M does not have the geodesic loops to infinity property, then there is a line in the universal cover $\widetilde{M}$.

Remark: *A line is a geodesic $\gamma : (-\infty, \infty) \to M$ such that $\operatorname{dist}(\gamma(s), \gamma(t)) = |s-t|$; a geodesic loop is a curve $\beta : [0,1] \to M$ that is a geodesic and $\beta(0) = \beta(1)$.*

Solution: Suppose $[\alpha] \in \pi_1(M)$ is a loop that M has no geodesic loops to infinity with respect to α, K. Suppose α is based at x_0. Let K be a compact subset $K \subset B_R(x_0) \subset M$. Let us choose $x_i \in M$ with $\operatorname{dist}(x_0, x_i) > R$. Minimize curves passing through x_i in the homotopy class $[\alpha]$ to get a geodesic loop γ_i that is based at x_i. Because M has no geodesic loops to infinity with respect to α, γ_i intersects with K; let $y_i \in K \cap \gamma_i$.

Now we go to the universal cover $\widetilde{M}$, and consider the lift $\tilde{\gamma}_i$ of γ_i, such that $\tilde{\gamma}_i$ connects $\tilde{x}_i$ and $[\alpha]\tilde{x}_i$ in the universal cover. We assume $\tilde{y}_i$ is the lift of y_i lying on $\tilde{\gamma}_i$. Let us estimate the distance d_i between $\tilde{y}_i$ and $[\alpha]x_i$. Because $\tilde{\gamma}_i$ is a minimizing geodesic segment, we have $\operatorname{dist}(\tilde{y}_i, [\alpha]\tilde{x}_i)$ equals to the length of the geodesic line segment $\tilde{\gamma}_i$ from $\tilde{y}_i$ to $[\alpha]\tilde{x}_i$. This is exactly the length of the part of the geodesic loop that connects y_i and x_i. By the triangle inequality, $d_i \geq m_i - R$. Similarly, the distance e_i between $\tilde{y}_i$ and x_i satisfies the bound $e_i \geq m_i - R$.

Therefore, there is a geodesic starting from $\tilde{y}_i$ that extends to both directions with length longer than $m_i - R$. Notice that $y_i \in K$, so for any i, we can choose $\tilde{y}_i$ in some fixed compact domain of $\widetilde{M}$. Then as $m_i \to \infty$, we can pass to a subsequence of $\tilde{y}_i$ to get a limit $\tilde{y}_\infty$, and a line passing through this point.

Problem 5. A topological space X is called an *H-space* if there exist $e \in X$ and $\mu : X \times X \to X$ such that $\mu(e,e) = e$ and the maps $x \to \mu(e,x)$ and $x \to \mu(x,e)$ are both homotopic to the identity map.

(a) Show that the fundamental group of an H-space is Abelian.

(b) Show that the sphere S^{2022} is not an H-space.

Historic Remark: *"H" was suggested by Jean-Pierre Serre in recognition of the contributions in Topology by Heinz Hopf.*

Solution:

(a) Let $[f]$ and $[g]$ be two elements in the fundamental group of X. We may assume $f : [0,1] \to X$ and $g : [0,1] \to X$ are both continuous maps with $f(0) = f(1) = g(0) = g(1) = e$.

Now we define a map $F : [0,1] \times [0,1] \to X$ by $F(x,y) = \mu(f(x), g(y))$. Then $F(\cdot, 0)$ is homotopic to f and $F(0,\cdot)$ is homotopic to g. It is clear that

$$h(s,t) = \begin{cases} F((1-t)2s, t\cdot 2s) & s \in [0, \frac{1}{2}], \\ F\big(t\cdot 2(s-\frac{1}{2}), (1-t)\cdot 2(s-\frac{1}{2})\big) & s \in [\frac{1}{2}, 1] \end{cases}$$

is a homotopy from a curve representing $[f]\cdot[g]$ to a curve representing $[g]\cdot[f]$. Therefore $[f]\cdot[g] = [g]\cdot[f]$, and hence $\pi_1(X)$ is Abelian.

(b) We will show that S^{2n} is not a H-space. In the following we consider $\mathbf{R}$-coefficient cohomology. Suppose S^k is an H-space, then the map μ induces

$$\mu^* : H^*(S^k) \to H^*(S^k) \otimes H^*(S^k),$$

and for a generator $x \in H^k(S^k)$, $\mu^*(x) = 1\otimes x + x \otimes 1$ (to see this, one can consider the composition $X \hookrightarrow X\times X \xrightarrow{\mu} X$, where the inclusion is $a \to (a,e)$ or $a \to (e,a)$).

The universality of the cup product gives

$$\mu^*(x \cup x) = (1\otimes x + x\otimes 1) \cup (1 \otimes x + x \otimes 1).$$

The left hand side is clearly 0, and the right hand side is $(1 + (-1)^{k^2})x \otimes x$. Here, notice that $(a\otimes b)\cup(c\otimes d) = (-1)^{|\deg(b)||\deg(c)|}(a\cup c)\otimes(b\cup d)$. Thus, S^k being an H-space implies that k is odd.

Remark: *In fact, Adams' Hopf invariant one theorem shows that among all the spheres, only S^0, S^1, S^3, S^7 are H-spaces.*

Problem 6. A hypersurface $\Sigma \subset \mathbf{R}^{\mathbf{n+1}}$ is called a *shrinker* if it satisfies the equation

$$H(x) = \tfrac{1}{2}\langle x, \vec{n}\rangle.$$

Here H is the mean curvature, which is $-\langle \mathrm{tr}_A, \vec{n}\rangle$ where A is the second fundamental form, x is the position vector, and $\vec{n}$ is outer unit normal vector.

(a) Show that $S^n(\sqrt{2n})$, the sphere with radius $\sqrt{2n}$, is a shrinker.

(b) Show that any compact shrinker without boundary must intersect with $S^n(\sqrt{2n})$.

Solution:

(a) One can calculate that for $S^n(\sqrt{2n})$, $A = -\dfrac{1}{\sqrt{2n}} g\vec{n}$, hence $H = \dfrac{n}{\sqrt{2n}}$. Also, $x = \sqrt{2n}\vec{n}$, so $S^n(\sqrt{2n})$ satisfies the shrinker's equation.

(b) Suppose Σ is a closed shrinker. On any hypersurface, $\nabla_\Sigma x = I$, where I is the $(n+1)\times(n+1)$ matrix that is the identity on $T_x\Sigma \otimes T_x\Sigma$ and vanishes elsewhere, $\Delta_\Sigma x = -H\vec{n}$, so

$$\Delta_\Sigma |x|^2 = 2\langle \nabla_\Sigma x, \nabla_\Sigma x\rangle + 2\langle \Delta_\Sigma x, x\rangle = 2n - 2\langle x, \vec{n}\rangle^2.$$

Consider $x_{\max}$ such that $|x|^2$ attains the maximum, and $x_{\min}$ such that $|x|^2$ attains the minimum. First let us consider $x_{\min} \neq 0$. Differentiating $|x|^2$ shows that $x_{\max}$ and $x_{\min}$ are normal to the tangent hyperplane, and $\langle x_\cdot, \vec{n}\rangle^2 = |x_\cdot|^2$ for $\cdot = \max$ or $\cdot = \min$. Then $\Delta_\Sigma|x|^2 \leq 0$ at $x_{\max}$, hence $2n - \langle x_{\max}, \vec{n}\rangle^2 \leq 0$. This implies that $|x_{\max}|^2 \geq 2n$. Similarly, $\Delta_\Sigma |x|^2 \geq 0$ at $x_{\min}$, and $|x_{\min}|^2 \leq 2n$. Finally, if $x_{\min} = 0$, then it is clear $|x_{\min}|^2 \leq 2n$. Therefore Σ must intersect $S^n(\sqrt{2n})$.