S.-T. Yau College Student Mathematics Contests 2010

Algebra, Number Theory and Combinatorics

Team

(Please select 5 problems to solve)

1. For a real number r, let $[r]$ denote the maximal integer less or equal than r. Let a and b be two positive irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. Show that the two sequences of integers $[ax]$, $[bx]$ for $x = 1, 2, 3, \cdots$ contain all natural numbers without repetition.

2. Let $n \geq 2$ be an integer and consider the Fermat equation

$$X^n + Y^n = Z^n, \qquad X, Y, Z \in \mathbb{C}[t].$$

Find all nontrivial solution (X, Y, Z) of the above equation in the sense that X, Y, Z have no common zero and are not all constant.

3. Let $p \geq 7$ be an odd prime number.

(a) Evaluate the rational number $\cos(\pi/7) \cdot \cos(2\pi/7) \cdot \cos(3\pi/7)$.

(b) Show that $\prod_{n=1}^{(p-1)/2} \cos(n\pi/p)$ is a rational number and determine its value.

4. For a positive integer a, consider the polynomial

$$f_a = x^6 + 3ax^4 + 3x^3 + 3ax^2 + 1.$$

Show that it is irreducible. Let F be the splitting field of f_a. Show that its Galois group is solvable.

5. Prove that a group of order 150 is not simple.

6. Let $V \cong \mathbb{C}^2$ be the standard representation of $SL_2(\mathbb{C})$.

(a) Show that the n-th symmetric power $V_n = \operatorname{Sym}^n V$ is irreducible.

(b) Which V_n appear in the decomposition of the tensor product $V_2 \otimes V_3$ into irreducible representations?