S.-T. Yau College Student Mathematics Contests 2021

Geometry and Topology

Solve every problem.

Problem 1.

(a) Show that $\mathbf{P^{2n}}$ can not be the boundary of a compact manifold.

(b) Show that $\mathbf{P^3}$ is the boundary of some compact manifold.

Solution:

(a) Suppose M is a compact manifold and ∂M is its boundary. We can glue together two copies of M, say M_1, M_2, to get a closed manifold $\widetilde{M}$. From the Mayer-Vietoris long exact sequence for the triad $(\widetilde{M}; M_1, M_2)$, we have the identity

$$\chi(\widetilde{M}) = 2\chi(M) - \chi(\partial M),$$

where χ is the Euler characteristic. If the dimension of ∂M is even, then the dimension of M is odd, and so is the dimension of $\widetilde{M}$. By Poincate duality, $\chi(\widetilde{M}) = 0$. So $\chi(\partial M)$ has to be an even number. However, $\mathbf{RP^{2n}}$ has odd Euler characteristic. Thus $\mathbf{RP^{2n}}$ can not be the boundary of a compact manifold.

(b) Since $\mathbf{RP^3}$ is diffeomorphic to SO(3), it is actually the circle bundle on S^2 in the tangent bundle of S^2. Thus $\mathbf{RP^3}$ is the boundary of the disk bundle of S^2 in the tangent bundle of S^2.

Problem 2. Suppose M is a noncompact, complete n-dimensional manifold, and suppose there is an open subset $U \subset M$ and an open set $V \subset \mathbf{R^n}$ such that $M \backslash U$ is isomorphic to $\mathbf{R^n} \backslash V$. If $\mathrm{Ric}M \geq 0$, show that M is isometric to $\mathbf{R^n}$.

Solution: Without loss of generality, we may assume $V = B_R(0)$ for some $R > 0$. Let $p \in M$ be a point. If $\mathrm{Ric}M \geq 0$, by the Bishop-Gromov inequality, we know $\dfrac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r^n)}$ is non-increasing. Here, B_r^n is the standard Euclidean ball in $\mathbf{R^n}$ with radius r. On one hand, as $r \to 0$, because M is a smooth manifold, we have $\lim\limits_{r\to 0} \dfrac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r^n)} = 1$; one the other hand, when $r \to \infty$, because $M \backslash U$ is isomorphic to $\mathbf{R^n} \backslash B_R(0)$, we also have $\lim\limits_{r\to\infty} \dfrac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r^n)} = 1$. As a consequence, $\dfrac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r^n)}$ is a constant 1 for all $r > 0$. Then by the rigidity case of Bishop-Gromov theorem, M is isometric to $\mathbf{R^n}$.

Remark: *Some students may want to use the Cheeger-Gromoll splitting theorem to show $M \cong N \times \mathbf{R}$, then conclude that $M \cong \mathbf{R^n}$. To my knowledge, it is actually hard to show that one can find a straight line in M. In fact, for a straight line in $\mathbf{R^n} \backslash V$, its corresponding line in M may not be straight, because there could be a shorter path going through U which connects two points on the line.*

Problem 3. Compute all the homotopy groups of the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$, $n \geq 2$.

Solution: In the following homotopy groups we always assume that we have fixed a base point.

Because T^n is connected, $\pi_0(T^n)$ is a trivial group.

$\pi_1(T^n)$ is the fundamental group of T^n. Because the fundamental group of a product space is just the product of each fundamental group, and the fundamental group of S^1 is $\mathbf{Z}$, so $\pi_1(T^n) = \mathbf{Z^n}$.

The universal cover of T^n is $\mathbf{R^n}$, which is contractible. So for all $k \geq 2$, $\pi_k(T^n) \cong \pi_k(\mathbf{R^n}) = 0$, which is the trivial group.

Problem 4. Consider the upper half space $\mathbf{H^3} = \{(x, y, z) \mid z > 0\}$ equipped with hyperbolic metric $g = \dfrac{dx^2 + dy^2 + dz^2}{z^2}$. Let P be the surface defined by $\{z = x \tan\alpha,\ z > 0\}$ for some $\alpha \in (0, \frac{\pi}{2})$. Compute the mean curvature of P.

Solution: We use ∂_x, ∂_y and ∂_z to denote the vector fields on $\mathbf{H^3}$ induced from $\mathbf{R^3}$. Then we can compute the Christoffel symbols

$$\begin{cases} \Gamma^i_{zi} = -z^{-1} & \\ \Gamma^z_{jj} = z^{-1}, & j \neq z \\ \Gamma^k_{ij} = 0, & \text{other cases.} \end{cases}$$

So the covariant derivatives are

$$\begin{cases} \nabla_{\partial_x}\partial_x = z^{-1}\partial_z \\ \nabla_{\partial_y}\partial_y = z^{-1}\partial_z \\ \nabla_{\partial_z}\partial_z = -z^{-1}\partial_z \\ \nabla_{\partial_z}\partial_x = \nabla_{\partial_x}\partial_z = -z^{-1}\partial_x \\ \nabla_{\partial_z}\partial_y = \nabla_{\partial_y}\partial_z = -z^{-1}\partial_y \\ \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = 0 \end{cases}$$

Now consider the surface P parametrized by $F : (u, v) \longrightarrow (u, v, u\tan\alpha)$. Then at any fixed point the tangent space is spanned by

$$F_u = \partial_x + \tan\alpha\partial_z,\ F_v = \partial_y,$$

with the metric

$$g_{uv} = z^{-2}\begin{pmatrix} 1 + \tan^2\alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

We can also find a unit normal vector

$$\mathbf{n} = \frac{z}{\sqrt{1 + \tan^2\alpha}}(\tan\alpha\partial_x - \partial_z).$$

Next, we can compute that

$$\begin{aligned} \nabla_{F_u}F_u &= (1 - \tan^2\alpha)z^{-1}\partial_z - 2\tan\alpha z^{-1}\partial_x, \\ \nabla_{F_v}F_u &= -\tan\alpha z^{-1}\partial_y, \\ \nabla_{F_v}F_v &= z^{-1}\partial_z. \end{aligned}$$

Moreover,

$$\begin{aligned} \langle\nabla_{F_u}F_u, \mathbf{n}\rangle &= -\tfrac{1}{z^2\sqrt{1+\tan^2\alpha}}(1 + \tan^2\alpha), \\ \langle\nabla_{F_v}F_u, \mathbf{n}\rangle &= 0, \\ \langle\nabla_{F_v}F_v, \mathbf{n}\rangle &= -\tfrac{1}{z^2\sqrt{1+\tan^2\alpha}}. \end{aligned}$$

So the mean curvature is

$$H = -g^{ij}\langle\nabla_{F_i}F_j, \mathbf{n}\rangle = \frac{2}{\sqrt{1 + \tan^2\alpha}} = 2\cos\alpha.$$

Remark: *There are different conventions for the definition of mean curvature, so the final answer could be* $\cos\alpha$*,* $-2\cos\alpha$*, or* $-\cos\alpha$*, depending on the choice of definitions.*

Problem 5. Suppose M is a compact 2-dimensional Riemannian manifold without boundary, with positive sectional curvature. Show that any two compact closed geodesics on M must intersect with each other.

Solution: We prove by contradiction. Suppose there exist two compact closed geodesics γ_1 and γ_2 that do not intersect with each other. Then we can find $p \in \gamma_1$ and $q \in \gamma_2$ such that the distance between p, q is the shortest distance among all pairs of points on γ_1 and γ_2. Let $\tilde{\gamma} : [a, b] \longrightarrow M$ be a length parametrized geodesic connecting $\tilde{\gamma}(a) = p$ and $\tilde{\gamma}(b) = q$, whose length realizes this shortest distance. Let ℓ be the length functional of curves. By the first variational formula,

$$\delta\ell(\tilde{\gamma}) = 0.$$

Namely, if V is a normal variational vector field along $\tilde{\gamma}$, suppose $\tilde{\gamma}_s$ is a family of curves generating this variational vector field, then

$$0 = \left.\frac{d\ell(\tilde{\gamma}_s)}{ds}\right|_{s=0} = -\left\langle V(a), \dot{\tilde{\gamma}}(a)\right\rangle + \left\langle V(b), \dot{\tilde{\gamma}}(b)\right\rangle .$$

As a consequence, we know that $\dot{\tilde{\gamma}}(a)$ is perpendicular to γ_1 at p and $\dot{\tilde{\gamma}}(b)$ is perpendicular to γ_2 at q.

Next we consider the second variational formula. Suppose X is a vector field along $\tilde{\gamma}$, where $|X(a)| = 1$ and $X(a)$ is perpendicular to $\tilde{\gamma}(a)$, and $X(t)$ is defined by parallel transport along $\tilde{\gamma}$ for $a < t \leq b$. Suppose $\tilde{\gamma}_s$ is a family of curves that generate X, then

$$0 \leq \left.\frac{d^2\ell(\tilde{\gamma}_s)}{ds^2}\right|_{s=0} = \int_a^b -R(\dot{\tilde{\gamma}}, X, X, \dot{\tilde{\gamma}})dt + \langle \nabla_{X(a)}X(a), \dot{\tilde{\gamma}}(a)\rangle - \langle \nabla_{X(b)}X(b), \dot{\tilde{\gamma}}(b)\rangle.$$

Notice that γ_1 and γ_2 are geodesics and $X(a)$, $X(b)$ are both unit vectors in the direction of γ_1, γ_2 at p, q respectively, so $\nabla_{X(a)}X(a) = 0$ and $\nabla_{X(b)}X(b) = 0$, and as a consequence

$$0 \leq \left.\frac{d^2\ell(\tilde{\gamma}_s)}{ds^2}\right|_{s=0} = \int_a^b -R(\dot{\tilde{\gamma}}, X, X, \dot{\tilde{\gamma}})dt = \int_a^b -\sec(\dot{\tilde{\gamma}}, X)dt < 0.$$

This is a contradiction.

Problem 6. Suppose Σ is a smooth compact embedded hypersurface (*i.e.* a codimension 1 submanifold) without boundary in $\mathbf{R^n}$ for $n \geq 3$. Show that Σ is orientable.

Solution: We first claim that it suffices to show Σ has a trivial normal bundle in $\mathbf{R^n}$. In fact, the trivial bundle has the splitting $\mathbf{R^n} \times \Sigma = T\Sigma \oplus N\Sigma$, so the first Stiefel-Whitney class of the bundles satisfies

$$0 = w_1(T\Sigma) + w_1(N\Sigma).$$

If the line bundle of Σ is trivial, we must have $w_1(N\Sigma) = 0$, therefore $w_1(T\Sigma) = 0$. This is equivalent to Σ being orientable.

Thus it remains to show that Σ has a trivial normal bundle. We prove by contradiction. We can view the tubular neighbourhood $\mathcal{T}$ of Σ as a part of $N\Sigma$. If Σ has a non-trivial normal bundle, then there exists a closed curve γ in $\mathcal{T}$ that only intersects Σ at a single point transversely. Consider a smoothly embedded disk D bounded by γ that intersects Σ transversely. Then the intersection of D and Σ consists of finitely many smooth curves whose endpoints lie on the boundary $\partial D = \gamma$. This implies that γ intersects Σ at an even number of points, which is a contradiction.