GROUP TEST

S.-T YAU COLLEGE MATH CONTESTS 2012

Algebra and Number Theory

Please solve 5 out of the following 6 problems.

1. Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be complex numbers such that $a_i+b_j \neq 0$ for all $i, j = 1, \cdots, n$. Define $c_{ij} := \dfrac{1}{a_i + b_j}$ for all $i, j = 1, \cdots, n$, and let C be the $n \times n$ determinant with entries c_{ij}. Prove that

$$det(C) = \frac{\prod_{1 \leq i < j \leq n}(a_i - a_j)(b_i - b_j)}{\prod_{1 \leq i, j \leq n}(a_i + b_j)}.$$

2. Recall that $\mathbb{F}_7$ is the finite field with 7 elements, and $GL_3(\mathbb{F}_7)$ is the group of all invertible 3×3 matrices with entries in $\mathbb{F}_7$.

(1) Find a 7-Sylow subgroup P_7 of $GL_3(\mathbb{F}_7)$.

(2) Determine the normalizer subgroup N of the 7-Sylow subgroup you found in (a).

(3) Find a 2-Sylow subgroup of $GL_3(\mathbb{F}_7)$.

3. Let V be a finite dimensional vector space with a positive definite quadratic form $(-, -)$. Let $O(V)$ denote the orthogonal group:

$$O(V) = \{g \in GL(V) : \qquad (gx, gy) = (x, y), \quad \forall x, y \in V\}.$$

For any non-zero $v \in V$, let s_v denote the reflection on V:

$$s_v(w) = w - 2\frac{(v, w)}{(v, v)}v.$$

(1) Show that $s_v \in O(V)$;

(2) Show that if v and w are vectors in V with $\|v\| = \|w\|$,then there is either a reflection or product of two reflections that takes v into w;

(3) Deduce that every element of the orthogonal group of V can be written as the product of at most $2 \dim V$ reflections.

4. Consider the real Lie group $SL_2(\mathbb{R})$ of 2 by 2 matrices of determinant one. Compute the fundamental group of $SL_2(\mathbb{R})$ and describe the Lie group structure on the universal covering

$$\widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R}).$$

5. Let $f \in \mathbb{C}[x, y, z]$ be an irreducible homogenous polynomial of degree $d > 0$. For each integer $n \geq d$, define

$$P(n) = \dim_{\mathbb{C}} \mathbb{C}[x, y, z]_n / f \cdot \mathbb{C}[x, y, z]_{n-d}$$

where $\mathbb{C}[x, y, z]_d$ is the subspace of homogenous polynomials of degree n. Show there are constants c such that for n sufficiently large,

$$P(n) = dn + c.$$

6. Let p be an odd prime and $\mathbb{Z}_p$ the p-adic integer which can be defined as the projective limit of $\mathbb{Z}/p^n\mathbb{Z}$ and let $\mathbb{Q}_p$ be its fractional field. Let $\mathbb{Z}_p^\times$ denote the group of invertible elements in $\mathbb{Z}_p$ which is also the projective limit of $(\mathbb{Z}/p^n\mathbb{Z})^\times$.

(1) For any integer a is not divisible by p, show that the sequence $(a^{p^n})_n$ convergent to an element $\omega(a) \in \mathbb{Z}_p$ satisfying

$$\omega(a)^{p-1} = 1, \qquad \omega(a) \equiv a \pmod p.$$

Moreover, $\omega(a)$ depends only on $a \mod p$.

(2) Define a logarithmic function log on $1 + p\mathbb{Z}_p$ by usual formula:

$$\log(1 + px) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{p^n}{n} x^n.$$

Show that the logarithmic function is convergent and define an isomorphism

$$1 + p\mathbb{Z}_p \to p\mathbb{Z}_p.$$

Moreover, on the dense subgroup $\log(1 + p)\mathbb{Z}$, the inverse is given by

$$\log(1 + p) \cdot x \mapsto (1 + p)^x, \qquad \forall x \in \mathbb{Z}.$$

(3) Deduce from above that $\mathbb{Z}_p^\times \simeq \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$.