S.-T. Yau College Student Mathematics Contests 2024

Geometry and Topology

Problem 1. Let $n > 1$ be a positive integer.

(i) Does there exist a map $f : S^{2n} \to \mathbb{CP}^n$ with $\deg(f) \neq 0$? Construct an example or disprove it.

(ii) Does there exist a map $f : \mathbb{CP}^n \to S^{2n}$ with $\deg(f) \neq 0$? Construct an example or disprove it.

Problem 2. Let $\Sigma \subset \mathbb{R}^3$ be an embedded surface in $\mathbb{R}^3$. A surface is called minimal if, for any $p \in \Sigma$, we have $\kappa_1(p) + \kappa_2(p) = 0$, where $\kappa_1(p)$ and $\kappa_2(p)$ are the two principal curvatures at p. Prove that if Σ is closed, then Σ cannot be minimal.

Problem 3. Let M be a closed, simply connected 6-dimensional manifold. Suppose $H_2(M) = \mathbb{Z}_2$. Prove that the Euler characteristic $\chi(M) \neq -1$.

Problem 4. Let (M, g) be a closed oriented n-dimensional Riemannian manifold. Let $p \in M$ and Ric_p be the Ricci curvature tensor at p, $_p$ be the scalar curvature at p which is defined to be $S_p := \frac{1}{n}\mathrm{Tr}_g(\mathrm{Ric}_p)$. Prove that the scalar curvature $S(p)$ at $p \in M$ is given by

$$S_p = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathrm{Ric}_p(V, V) dS^{n-1},$$

where ω_{n-1} is the area of the unit sphere S^{n-1} in T_pM, $V \in S^{n-1}$ are unit vector fields, and dS^{n-1} is the area element on S^{n-1}.

Problem 5. Let S^n be the n-dimensional sphere with $n \geq 2$, and let G be a finite group that acts freely on S^n. Suppose G is non-trivial. Then,

(i) Compute the homotopy groups of the quotient space $\pi_i(S^n/G)$ for $0 \leq i \leq n$.

(ii) Suppose n is even. Prove that G is isomorphic to $\mathbb{Z}_2$.

(iii) Suppose n is odd. Show that G cannot be isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for p a prime number.

Problem 6. Let M be a closed oriented Riemannian manifold, where g_t is a family of smooth Riemannian metrics smoothly depending on $t \in (-\epsilon, \epsilon)$. Suppose there exists a family of eigenfunctions f_t and eigenvalues λ_t smoothly depending on t such that

$$\Delta_{g_t} f_t = \lambda_t f_t,$$

where Δ_{g_t} is the Laplace-Beltrami operator defined using the Riemannian metric g_t. Additionally, assume that f_0 is not a constant function. We define $\dot{\lambda} := \frac{d}{dt}|_{t=0}\lambda_t$ and $\dot{\Delta} := \frac{d}{dt}|_{t=0}\Delta_{g_t}$. Prove the following:

(i) As λ_0 is an eigenvalue of Δ_{g_0}, let $V_{\lambda_0} := \mathrm{Ker}(\Delta_{g_0} - \lambda_0)$ be the eigenspace of λ_0, and let $\Pi : L^2(M, g_0) \to V_{\lambda_0}$ be the orthogonal projection onto the eigenspace. Prove that $\dot{\lambda}$ is an eigenvalue of the operator $\Pi \circ \Delta' : V_{\lambda_0} \to V_{\lambda_0}$.

(ii) Let $\varphi_t : M \to M$ be a 1-parameter family of diffeomorphisms of M and assume $g_t = \varphi_t^* g_0$. Prove that $\dot{\lambda} = 0$.