S.-T. Yau College Student Mathematics Contests 2021

Analysis and Differential Equations

Solve every problem.

Problem 1. Prove that $f(x) \equiv 0$ is the only solution in $L^2(\mathbf{R^n})$ such that

$$\triangle f = 0.$$

Problem 2. Let $X \subset C([0,1])$ be a finite dimensional linear subspace of the space of real-valued continuous functions on $[0,1]$. Show that, for a sequence of functions $\{f_k\}_{k \geqslant 1} \subset X$, if it converges pointwise, it converges uniformly.

Problem 3.

(a) For $f \in L^1(\mathbf{R^n})$, $g \in L^\infty(\mathbf{R^n})$, show that their convolution $f * g$ is a well-defined continuous function.

(b) Let $E \subset \mathbf{R^n}$ be a Lebesgue measurable set with Lebesgue measure $m(E) > 0$. Prove that

$$E - E := \left\{ x - y \,\middle|\, x \in E, y \in E \right\}$$

contains an open neighborhood of $0 \in \mathbf{R^n}$.

Problem 4. Assume that P is a polynomial with complex coefficients. Prove that there exists infinitely many solutions of the following equations on $\mathbf{C}$:

$$e^z = P(z).$$

Problem 5. Let f be a bounded holomorphic function defined on $B = \left\{ z \,\middle|\, 0 < \mathrm{Re}(z) < 1 \right\}$ that can be extended as a continuous function on $\overline{B}$. Let

$$A_0 = \sup_{\mathrm{Re}(z)=0} |f(z)| > 0, \quad A_0 = \sup_{\mathrm{Re}(z)=1} |f(z)| > 0.$$

Prove that for all $z \in B$, we have

$$|f(z)| \leqslant (A_0)^{1-\mathrm{Re}(z)} (A_1)^{\mathrm{Re}(z)}.$$

Problem 6. Assume that $\Omega \subset \mathbf{R^n}$ is a bounded domain with smooth boundary. Prove that there exists a positive constant ε_0 so that for all real numbers $\varepsilon < \varepsilon_0$, for all $f \in L^2(\Omega)$, there exist a unique $u \in H_0^1(\Omega)$ so that

$$-\triangle u + \varepsilon \sin(u) = f$$

in the sense of distributions.