S.-T. Yau College Student Mathematics Contests 2017

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let M be a smooth, compact, oriented n-dimensional manifold. Suppose that the Euler characteristic of M is zero. Show that M admits a nowhere vanishing vector field.

2. Let $S^2 \xleftarrow{q_1} S^2 \vee S^2 \xrightarrow{q_2} S^2$ be the maps that crush out one of the two summands. Let $f : S^2 \to S^2 \vee S^2$ be a map such that $q_i \circ f : S^2 \to S^2$ is a map of degree d_i. Compute the integral homology groups of $(S^2 \vee S^2) \cup_f D^3$. Here D^3 is the unit solid ball with boundary S^2.

3. Let X and Y be smooth vector fields on a smooth manifold. Prove that the Lie derivative satisfies the identity

$$L_X Y = [X, Y].$$

4. State and prove the Liouville formula for the geodesic curvature κ_g along a regular curve on a smooth surface in $\mathbb{R}^3$.

5. On a Riemannian manifold, let F be the set of smooth functions f on M with $|\mathrm{grad} f| \leq 1$. For any x, y in the manifold, show that

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in F\}.$$

6. Let M be an n-dimensional oriented closed minimal submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Denote by K_M the sectional curvature of M. Prove that if $K_M > \dfrac{p-1}{2p-1}$, then M is the great sphere S^n.