S.-T. Yau College Student Mathematics Contests 2010

Applied Math., Computational Math., Probability and Statistics

Team

(Please select 5 problems to solve)

1. Let $X_1, \cdots, X_n$ be independent and identically distributed random variables with continuous distribution functions $F(x_1), \cdots, F(x_n)$, respectively. Let $Y_1 < \cdots < Y_n$ be the order statistics of $X_1, \cdots, X_n$. Prove that $Z_j = F(Y_j)$ has the beta $(j, n-j+1)$ distribution $(j = 1, \cdots, n)$.

2. Let $X_1, \cdots, X_n$ be i.i.d. random variable with a continuous density f at point 0. Let

$$Y_{n,i} = \frac{3}{4b_n}(1 - X_i^2/b_n^2)I(|X_i| \le b_n).$$

Show that

$$\frac{\sum_{i=1}^n (Y_{n,i} - EY_{n,i})}{(b_n \sum_{i=1}^n Y_{n,i})^1/2} \xrightarrow{L} N(0, 3/5),$$

provided $b_n \to 0$ and $nb_n \to \infty$.

3. Let $X_1, \cdots, X_n$ be independently and indentically distributed random variables with $X_i \sim N(\theta, 1)$. Suppose that it is known that $|\theta| \le \tau$, where τ is given. Show

$$\min_{a_1, \cdots, a_{n+1}} \sup_{|\theta| \le \tau} E(\sum_{i=1}^n a_i X_i + a_{n+1} - \theta)^2 = \frac{\tau^2 n^{-1}}{\tau^2 + n^{-1}}.$$

Hint: Carefully use the sufficiency principle.

4. The rules for "1 and 1" foul shooting in basketball are as follows. The shooter gets to try to make a basket from the foul line. If he succeeds, he gets another try. More precisely, he make 0 baskets by missing the first time, 1 basket by making the first shot and xsmissing the second one, or 2 baskets by making both shots.

Let n be a fixed integer, and suppose a player gets n tries at " 1 and 1" shooting. Let N_0, N_1, and N_2 be the random variables recording the number of times he makes 0, 1, or 2 baskets, respectively. Note that $N_0 + N_1 + N_2 = n$. Suppose that shots are independent Bernoulli trails with probability p for making a basket.

(a) Write down the likelihood for (N_0, N_1, N_2).

(b) Show that the maximum likelihood estimator of p is

$$\hat{p} = \frac{N_1 + 2N_2}{N_0 + 2N_1 + 2N_2}.$$

(c) Is $\hat{p}$ an unbiased estimator for p? Prove or disprove. (Hint: $E\hat{p}$ is a polynomial in p, whose order is higher than 1 for $p \in (0,1)$.)

(d) Find the asymptotic distribution of $\hat{p}$ as n tends to ∞.

5. When considering finite difference schemes approximating partial differential equations (PDEs), for example, the scheme

$$u_j^{n+1} = u_j^n - \lambda(u_j^n - u_{j-1}^n) \tag{1}$$

where $\lambda = \dfrac{\Delta t}{\Delta x}$, approximating the PDE

$$u_t + u_x = 0, \tag{2}$$

we are often interested in stability, namely

$$||u^n|| \le C||u^0||, \qquad n\Delta t \le T \tag{3}$$

for a constant $C = C(T)$ independent of the time step Δt and the spatial mesh size Δx. Here $||\cdot||$ is a given norm, for example the L^2 norm or the L^∞ norm, of the numerical solution vector $u^n = (u_1^n, u_2^n, \cdots, u_N^n)$. The mesh points are $x_j = j\Delta x$, $t^n = n\Delta t$, and the numerical solution u_j^n approximates the exact solution $u(x_j, t^n)$ of the PDE (2) with a periodic boundary condition.

(i) Prove that the scheme (1) is stable in the sense of (3) for both the L^2 norm and the L^∞ norm under the time step restriction $\lambda \le 1$.

(ii) Since the numerical solution u^n is in a finite dimensional space, Student A argues that the stability (3), once proved for a specific norm $||\cdot||_a$, would also automatically hold for any other norm $||\cdot||_b$. His argument is based on the equivalency of all norms in a finite dimensional space, namely for any two norms $||\cdot||_a$ and $||\cdot||_b$ on a finite dimensional space W, there exists a constant $\delta > 0$ such that

$$\delta||u||_b \le ||u||_a \le \frac{1}{\delta}||u||_b.$$

Do you agree with his argument? If yes, please give a detailed proof of the following theorem: If a scheme is stable, namely (3) holds for one particular norm (e.g. the L^2 norm), then it is also stable for any other norm. If not, please explain the mistake made by Student A.

6. We have the following 3 PDEs

$$u_t + Au_x = 0, \tag{4}$$

$$u_t + Bu_x = 0, \tag{5}$$

$$(6) \qquad u_t + Cu_x = 0, \qquad C = A + B.$$

Here u is a vector of size m and A and B are $m \times m$ real matrices. We assume $m \geq 2$ and both A and B are diagonalizable with only real eigenvalues. We also assume periodic initial condition for these PDEs.

(i) Prove that (4) and (5) are both well-posed in the L^2-norm. Recall that a PDE is well-posed if its solution satisfies

$$||u(\cdot, t)|| \leq C(T)||u(\cdot, 0)||, \qquad 0 \leq t \leq T$$

for a constant $C(T)$ which depends only on T.

(ii) Is (6) guaranteed to be well-posed as well? If yes, give a proof; if not, give a counter example.

(iii) Suppose we have a finite difference scheme

$$u^{n+1} = A_h u^n$$

for approximating (4) and another scheme

$$u^{n+1} = B_h u^n$$

for approximating (5). Suppose both schemes are stable in the L^2-norm, namely (3) holds for both schemes. If we now form the splitting scheme

$$u^{n+1} = B_h \, A_h u^n$$

which is a consistent scheme for solving (6), is this scheme guaranteed to be L^2 stable as well? If yes, give a proof; if not, give a counter example.