S.-T. Yau College Student Mathematics Contests 2014

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Compute the fundamental and homology groups of the wedge sum of a circle S^1 and a torus $T = S^1 \times S^1$.

2. Given a properly discontinuous action $F : G \times M \to M$ on a smooth manifold M, show that M/G is orientable if and only if M is orientable and $F(g, \cdot)$ preserves the orientation of M. Use this statement to show that the Möbius band is not orientable and that $\mathbb{R}P^n$ is orientable if and only if n is odd.

3. (a) Consider the space Y obtained from $S^2 \times [0, 1]$ by identifying $(x, 0)$ with $(-x, 0)$ and also identifying $(x, 1)$ with $(-x, 1)$, for all $x \in S^2$. Show that Y is homeomorphic to the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

(b) Show that $S^2 \times S^1$ is a double cover of the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

4. Prove that a bi-invariant metric on a Lie group G has nonnegative sectional curvature.

5. Let M be the upper half-plane $\mathbb{R}^2_+$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^k}.$$

For which values of k is M complete?

6. Given any nonorientable manifold M show the existence of a smooth orientable manifold $\overline{M}$ which is a double covering of M. Find $\overline{M}$ when M is $\mathbb{R}P^2$ or the Möbius band.