GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Probability and Statistics

Please solve 5 out of the following 6 problems.

1. Let (X_n) be a sequence of i.i.d. random variables.

1) Assume that each X_n satisfies the exponential distribution with parameter 1 (i.e. $P(X_n \geq x) = e^{-x}, x \geq 0$). Prove that

(a) $P(X_n > \alpha \log n, i.o.) = 0$, if $\alpha > 1$; $P(X_n > \alpha \log n, i.o.) = 1$, if $\alpha \leq 1$.

Here "i.o" stands for "infinitely often", and $A_n, i.o.$ stands $\limsup_{n\to\infty} A_n$.

(b) Let $L = \limsup_{n\to\infty}(X_n/\log n)$, then $P(L = 1) = 1$.

2) Assume that each X_n satisfies the Poisson distribution with parameter λ (i.e. $P(X_n = k) = \frac{\lambda^k}{k!}e^{-\lambda}, k = 0, 1, 2, \cdots.$) Put

$$L = \limsup_{n\to\infty}(X_n \log\log n/\log n).$$

Prove that $P(L = 1) = 1$.

2. Let X_i be i.i.d exponential r.v with rate one, $i \geq 1$. Let N be a geometric random variable with success probability p, $0 < p < 1$, i.e. $P(N = k) = (1-p)^{k-1}p, k = 1, 2, \cdots$, and independent of all X_i, $i \geq 1$. Find the distribution of $\sum_{i=1}^{N} X_i$.

3. Let X and Y be i.i.d real valued r.v's. Prove that $P(|X+Y| < 1) \leq 3P(|X - Y| < 1)$.

4. Suppose $S = X_1 + X_2 + \cdots + X_n$, a sum of independent random variables with X_i distributed Binomial$(1, p_i)$. Show that $\mathbb{P}(S\ even) = 1/2$ if and only if at least one p_i equals $1/2$.

5. Let B_θ denote the closed unit ball in $\mathbb{R}^2$ with center θ. Suppose $X_1, X_2, \cdots, X_n$ are independently and uniformly distributed on B_θ, for an unknown θ in $\mathbb{R}^2$. Denote that maximum likelihood estimator by $\hat{\theta}$. Show that $|\hat{\theta} - \theta| = O_p(1/n)$.

6. Suppose that $X_1, \cdots, X_n$ are a random sample from the Bernoulli distribution with probability of success p_1 and $Y_1, \cdots, Y_n$ be an independent random sample from the Bernoulli distribution with probability of success p_2.

(a) Derive the maximum likelihood ratio test statistic for

$$H_0 : p_1 = p_2 \longleftrightarrow H_1 : p_1 \neq p_2.$$

(Note: No simplification of the resulting test statistic is required. However, you need to give the asymptotic null.)

(b) Compute the asymptotic power of the test with critical region

$$|\sqrt{n}(\hat{p}_1 - \hat{p}_2)/\sqrt{2\hat{p}\hat{q}}| \geqslant z_{1-\alpha}$$

when $p_1 = p$ and $p_2 = p + n^{-1/2}\Delta$, where $\hat{p} = 0.5.\hat{p}_1 + 0.5\hat{p}_2$.