S.-T. Yau College Student Mathematics Contests 2018

Analysis and Differential Equations

Individual

Please solve the following 6 problems.

1. Let the sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $L^2(\mathbf{R}^d)$ satisfy that $\|f_n\|_{L^2} = 1$.

(1)Show that there exists a subsequence of function $\{f_{n_j}\}_{j=1}^{\infty}$ such that f_{n_j} converges weakly to some function f in $L^2(\mathbf{R}^d)$, i.e.

$$(f_{n_j}, g) \to (f, g)$$

for all $g \in L^2(\mathbf{R}^d)$.

(2)If $f_n \to f$ weakly in $L^2(\mathbf{R}^d)$, and $\|f_n\|_{L^2} \to \|f\|_{L^2}$ as $n \to \infty$. Show that $\|f_n - f\|_{L^2} \to 0$ as $n \to \infty$.

2. Let $f : U \to \mathbf{C}$ be a non-constant holomorphic function where $U \subset \mathbf{C}$ is the open set containing the closure $\overline{D}$ of the unit disk $D = \{z \in \mathbf{C} | |z| < 1\}$.

If $|f(z)| = 1$, for all $z \in \partial D$, Prove that $D \subset f(\overline{D})$.

3. Prove that if a sequence of harmonic function on the open disk converges uniformly on compact subset of the disk, then the limit is harmonic.

4. Let μ be a Borel measure on $\mathbf{R}^n$. Let $\rho > 0$, a fixed positive number, and $B_\rho(x) = \{y \in \mathbf{R}^n | d(x, y) < \rho\}$. For $x \in \mathbf{R}^n$, define a function:

$$\theta(x) : x \to \mu(\overline{B_\rho(x)})$$

1) Show that θ is upper semi-continuous, i.e. for every $x \in \mathbf{R}^n, \theta(x) \geq \limsup_{y \to x} \theta(y)$.

2) Give an example of a Borel measure μ, such that the function θ is not continuous.

5. Let g denote a smooth function on $\mathbf{R}^n$ with compact support. Let f denote the function given by the formula

$$f(x) = \frac{1}{n(n-1)\alpha(n)} \int\limits_{\mathbf{R}^n} \frac{1}{|x - y|^{n-2}} g(y) dy.$$

Here $\alpha(n)$ is volume of the unit ball in $\mathbf{R}^n$.

(a) Prove that the integral that defines f converges for each $x \in \mathbf{R}^n$.

(b) Prove that f is differentiable and that the gradient of f if given by the formula

$$\nabla f|_x = \frac{1}{n(n-1)\alpha(n)} \int\limits_{\mathbf{R}^n} \frac{1}{|x - y|^{n-2}} (\nabla g)|_y dy.$$

(c) Prove the f obeys $-\Delta f = g$ with Δ denoting $\dfrac{\partial^2}{\partial {x_1}^2} + \cdots + \dfrac{\partial^2}{\partial {x_n}^2}$.

6. If u is a positive harmonic function on $\mathbb{R}^n \setminus \{0\}$ $(n \geq 2)$, then exist constants $a \geq 0, b \geq 0$ such that

$$u(x) = a + b|x|^{2-n}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.