S.-T. Yau College Student Mathematics Contests 2017

Applied Math. and Computational Math.

Individual (5 problems)

1. The Chebyshev polynomial of the first kind is defined on $[-1, 1]$ by

$$T_n(x) = \cos(n \arccos x).$$

Prove: The envelope for the extremals of $T_{n+1}(x) - T_{n-1}(x)$ forms an ellipse.

2. Consider a fixed point iteration

$$x_n = g(x_{n-1}),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth function. Suppose this fixed point method does converge to a fixed point x^*. The Steffensen algorithm is an acceleration method to find x^* which reads

$$\hat{x}_n = x_{n-2} - \frac{(x_{n-1} - x_{n-2})^2}{x_n - 2x_{n-1} + x_{n-2}}.$$

or

$$x_{n+1} = G(x_n)$$

where

$$G(x) = x - \frac{(g(x) - x)^2}{g(g(x)) - 2g(x) + x}.$$

(a) Show that the Steffensen algorithm $\{x_k\}$ converges quadratically.
(b) Can you extend this method to two dimensions?

3. We consider a piecewise smooth function

$$f(x) = \begin{cases} f_1(x), & x \le 0, \\ f_2(x), & x > 0 \end{cases}$$

where $f_1(x)$ is a C^∞ function on $(-\infty, 0]$ and $f_2(x)$ is a C^∞ function on $[0, \infty)$, but $f_1(0) \neq f_2(0)$. Suppose $p(x)$ is a k-th degree polynomial ($k \geq 1$) interpolating $f(x)$ at $k+1$ equally-spaced grid points x_j, $j = 0, 1, 2, \cdots, k$ with $x_i < 0 < x_{i+1}$ for some i between 0 and $k-1$. Prove that, when the grid size $h = x_{j+1} - x_j$ is small enough, $p'(x) \neq 0$ for $x_i \leq 0 \leq x_{i+1}$, that is, $p(x)$ is monotone in the interval $[x_i, x_{i+1}]$. (**Hint: first prove the case when $f_1(x) = c_1$, $f_2(x) = c_2$ and $c_1 \neq c_2$ are two constants.**)

4. Let $b \in \mathbb{R}^n$. Suppose $A \in M_{n\times n}(\mathbb{R})$ and $B \in M_{n\times n}(\mathbb{R})$ are two $n \times n$ matrices. Let A to be non-singular.

(a) Consider the iterative scheme: $Ax^{k+1} = b - Bx^k$.
State and prove the necessary and sufficient condition for the iterative scheme to converge.

(b) Suppose the spectral radius of $A^{-1}B$ satisfies $\rho(A^{-1}B) = 0$. Prove that the iterative scheme converges in n iterations.

(c) Consider the following iterative scheme:

$$x^{(k+1)} = \omega_1 x^{(k)} + \omega_2(c_1 - Mx^{(k)}) + \omega_3(c_2 - Mx^{(k)}) + ... + \omega_k(c_{k-1} - Mx^{(k)})$$

where M is symmetric and positive definite, $\omega_1 > 1$, $\omega_2, ..., \omega_k > 0$ and $c_1, ..., c_{k-1} \in \mathbb{R}^n$. Deduce from (a) that the iterative scheme converges if and only if all eigenvalues of M (denote it as $\lambda(M)$) satisfies:

$$(\omega_1 - 1)/(\sum_{i=2}^{k} \omega_i) < \lambda(M) < (\omega_1 + 1)/(\sum_{i=2}^{k} \omega_i).$$

(d) Let A be non-singular. Now, consider the following system of iterative scheme (*):

$$Ax_1^{(k+1)} = b_1 - Bx_2^{(k)}, \quad Ax_2^{(k+1)} = b_2 - Bx_1^{(k)}$$

Find and prove the necessary and sufficient condition for the iterative scheme (*) to converge.

For the iterative scheme (**):

$$Ax_1^{(k+1)} = b_1 - Bx_2^{(k)}, \quad Ax_2^{(k+1)} = b_2 - Bx_1^{(k+1)}$$

Find and prove the necessary and sufficient condition for the iterative scheme (**) to converge. Compare the rate of convergence of the iterative schemes (*) and (**).

5. Consider the differential equation

$$-u'' + \alpha u = f, \ x \in (0, 1).$$

Here, prime denotes for d/dx and α is a constant. We consider a mixed boundary condition

$$u(0) = 0, \ u'(1) - bu(0) = 0.$$

This equation is approximated by a standard finite difference method:

$$\frac{-U_{j-1} + 2U_j - U_{j+1}}{h^2} + \alpha U_j = f_j, j = 1, ..., N - 1.$$

Here, N is the number of grid points, $h = 1/N$ is the mesh size, U_j is the approximate solution at $x_j := jh$, and $f_j = f(x_j)$. The noundary condition is approximated by

$$U_0 = 0, \ \frac{U_N - U_{N-1}}{h} - bU_N = 0.$$

The resulting linear system is $AU = F$ with

$$\begin{bmatrix} \beta & -1 & 0 & \cdots & & & \\ -1 & \beta & -1 & \cdots & & & \\ & & & \ddots & & & \\ & & & & -1 & \beta & -1 \\ & & & & 0 & -1 & 1-bh \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \\ U_N \end{bmatrix} = \begin{bmatrix} h^2 f_1 \\ h^2 f_2 \\ \vdots \\ h^2 f_{N-1} \\ 0 \end{bmatrix}$$

where $\beta = 2 + \alpha h^2$.

$$u_t + a u_x = 0, \ \ a > 0.$$

We discretize this PDE by For solving the following partial differential equation

$$u_t + f(u)_x = 0, \qquad 0 \le x \le 1 \tag{1}$$

where $f'(u) \ge 0$, with periodic boundary condition, we can use the following semi-discrete upwind scheme

$$\frac{d}{dt} u_j + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = 0, \qquad j = 1, 2, \cdots, N, \tag{2}$$

with periodic boundary condition

$$u_0 = u_N, \tag{3}$$

where $u_j = u_j(t)$ approximates $u(x_j, t)$ at the grid point $x = x_j = j\Delta x$, with $\Delta x = \frac{1}{N}$.

(a) Prove the following L^2 stability of the scheme

$$\frac{d}{dt} E(t) \le 0 \tag{4}$$

where $E(t) = \sum_{j=1}^{N} |u_j|^2 \Delta x$.

(b) Do you believe (4) is true for $E(t) = \sum_{j=1}^{N} |u_j|^{2p} \Delta x$ for *arbitrary* integer $p \ge 1$? If yes, prove the result. If not, give a counterexample.