S.-T. Yau College Student Mathematics Contests 2011

Analysis and Differential Equations

Individual

2:30–5:00 pm, July 9, 2011

(Please select 5 problems to solve)

1. a) Compute the integral: $\int_{-\infty}^{\infty} \frac{x \cos x dx}{(x^2+1)(x^2+2)}$,

b) Show that there is a continuous function $f : [0, +\infty) \to (-\infty, +\infty)$ such that $f \not\equiv 0$ and $f(4x) = f(2x) + f(x)$.

2. Solve the following problem:

$$\begin{cases} \frac{d^2u}{dx^2} - u(x) = 4e^{-x}, & x \in (0, 1), \\ u(0) = 0, \quad \frac{du}{dx}(0) = 0. \end{cases}$$

3. Find an explicit conformal transformation of an open set $U = \{|z| > 1\} \setminus (-\infty, -1]$ to the unit disc.

4. Assume $f \in C^2[a, b]$ satisfying $|f(x)| \le A, |f''(x)| \le B$ for each $x \in [a, b]$ and there exists $x_0 \in [a, b]$ such that $|f'(x_0)| \le D$, then $|f'(x)| \le 2\sqrt{AB} + D, \forall x \in [a, b]$.

5. Let $C([0, 1])$ denote the Banach space of real valued continuous functions on [0, 1] with the sup norm, and suppose that $X \subset C([0, 1])$ is a dense linear subspace. Suppose $l : X \to \mathbb{R}$ is a linear map (not assumed to be continuous in any sense) such that $l(f) \ge 0$ if $f \in X$ and $f \ge 0$. Show that there is a unique Borel measure μ on [0, 1] such that $l(f) = \int f d\mu$ for all $f \in X$.

6. For $s \ge 0$, let $H^s(T)$ be the space of L^2 functions f on the circle $T = \mathbb{R}/(2\pi\mathbb{Z})$ whose Fourier coefficients $\hat{f}_n = \int_0^{2\pi} e^{-inx} f(x)dx$ satisfy $\Sigma(1 + n^2)^s ||\hat{f}_n|^2 < \infty$, with norm $||f||_s^2 = (2\pi)^{-1}\Sigma(1 + n^2)^s|\hat{f}_n|^2$.

a. Show that for $r > s \ge 0$, the inclusion map $i : H^r(T) \to H^s(T)$ is compact.

b. Show that if $s > 1/2$, then $H^s(T)$ includes continuously into $C(T)$, the space of continuous functions on T, and the inclusion map is compact.