S.-T. Yau College Student Mathematics Contests 2022

Computational and Applied Mathematics

Solve every problem.

Problem 1. Consider $\{p_i(x)\}_{i=0}^{\infty}$, a family of orthogonal polynomials associated with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)w(x)\,dx, \quad w(x) > 0 \quad \text{for } x \in (-1,1),$$

where $p_i(x)$ is a polynomial of degree i. Let $x_0, x_1, \dots, x_n$ be the roots of $p_{n+1}(x)$. Construct an orthonormal basis in the subspace of the polynomials of degree no more than n such that, for any polynomial in this subspace, the coefficients of its expansion into the basis are equal to the scaled values of this polynomial at the nodes $x_0, x_1, \dots, x_n$.

Problem 2. Consider a 2D fixed point iteration of the form

$$x_{k+1} = f(x_k, y_k),\ y_{k+1} = g(x_k, y_k). \tag{1}$$

Assume that the vector-valued function $\vec{H}(x,y) = (f(x,y), g(x,y))^T$ is continuously-differentiable, and the infinity norm of the Jacobian matrix is less than 1 at a unique fixed point (x_∞, y_∞).

Now consider a new iteration:

$$x_{k+1} = f(x_k, y_k), \quad y_{k+1} = g(x_{k+1}, y_k). \tag{2}$$

Prove that iteration (2) is convergent, to the same fixed point as iteration (1), for the initial conditions sufficiently close to the fixed point.

Problem 3. Let $A \in \mathbf{R}^{\mathbf{m}\times\mathbf{m}}$ be a matrix with entries a_{ij} which satisfy

$$a_{ii} \geq \sum_{j\neq i} |a_{ij}| + 2, \quad a_{ii} \leq 7.$$

(a) Prove that A^{-1} exists.

(b) Prove that $\|A\|_\infty$ is the *max row sum* (of absolute values) of A.

(c) Find both a lower and upper bound for $\|A\|_\infty$.

(d) Now assume $A = A^T$. Find bounds for $\|A\|_2$ and $\|A^{-1}\|_2$.

Problem 4. Consider a system of ODE initial value problems of the form:

$$\frac{d}{dt}u = f(u), \quad u(0) = u_0.$$

Assume that $f(u)$ has the property that the forward Euler (FE) method:

$$U^{n+1} = U^n + kf(U^n),$$

satisfies

$$\|U^{n+1}\| \leq \|U^n\|$$

for some norm $\|\cdot\|$ and for all time-steps k, $0 < k \le k_{FE}$. Now consider the 2-stage Runge-Kutta method:

$$\begin{aligned} U^{(1)} &= U^n + k\beta_{10} f(U^n), \\ U^{n+1} &= \{\alpha_{20}U^n + k\beta_{20} f(U^n)\} + \left\{\alpha_{21}U^{(1)} + k\beta_{21} f(U^{(1)})\right\} \end{aligned}$$

where

$$\beta_{10} \ge 0, \quad \beta_{20} \ge 0, \quad \beta_{21} \ge 0, \quad \alpha_{20} \ge 0, \quad \alpha_{21} \ge 0, \quad \alpha_{20} + \alpha_{21} = 1.$$

(a) Prove that the above 2-stage Runge-Kutta method also satisfies the inequality:

$$\|U^{n+1}\| \le \|U^n\|$$

under some appropriate time-step restriction: $0 \le k \le k^*$, where you need to explicitly determine k^* in terms of k_{FE}.

(b) Explicitly determine the coefficients:

$$\beta_{10}, \quad \beta_{20}, \quad \beta_{21}, \quad \alpha_{20}, \quad \alpha_{21},$$

so that

(i) The method is second-order accurate; and

(ii) The maximum allowed time-step, k^*, is as large as possible.

Problem 5. Construct a third-order accurate Lax-Wendroff-type method for $u_t + au_x = 0$ ($a > 0$ is a constant) in the following way:

(a)
- Expand $u(t + k, x)$ in a Taylor series and keep the first four terms. Replace all time derivatives by spatial derivatives using the equation.
- Construct a cubic polynomial passing through the points $U^n_{j-2}, U^n_{j-1}, U^n_j, U^n_{j+1}$.
- Approximate the spatial derivatives in the Taylor series by the exact derivatives of the above constructed cubic polynomial.

(b) Verify that the truncation error is $O(k^3)$ if $h = O(k)$.

Problem 6. Suppose you have \$60K to invest and there are 3 investment options available. You must invest in multiples of \$10K. If d_i dollars are invested in investment i then you receive a net value (as the profit) of $r_i(d_i)$ dollars. For $d_i > 0$ we have

$$\begin{aligned} r_1(d_1) &= (7d_1 + 2) \times 10, \\ r_2(d_2) &= (3d_2 + 7) \times 10, \\ r_3(d_3) &= (4d_3 + 5) \times 10, \end{aligned}$$

and $d_1(0) = d_2(0) = d_3(0)$. All are measured in \$10K dollars. The objective is to maximize the net value of your investments. This can be formulated as a linear programming problem:

$$\max_{d_1,d_2,d_3} r_1(d_1) + r_2(d_2) + r_3(d_3),$$
$$\text{such that} \quad d_1 + d_2 + d_3 \leq 6,$$
$$d_i \geq 0 \quad i = 1, 2, 3 \quad \text{are integers.}$$