S.-T. Yau College Student Mathematics Contests 2010

Geometry and Topology

Team

(Please select 5 problems to solve)

1. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, and $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ the equator n-plane through the center of S^n. Let N be the north pole of S^n. Define a mapping $\pi: S^n \backslash \{N\} \rightarrow \mathbb{R}^n$ called the stereographic projection that takes $A \in S^n \backslash \{N\}$ into the intersection $A' \in \mathbb{R}^n$ of the equator n-plane $\mathbb{R}^n$ with the line which passes through A and N. Prove that the stereographic projection is a conformal change, and derive the standard metric of S^n by the stereographic projection.

2. Let M be a (connected) Riemannian manifold of dimension 2. Let f be a smooth non-constant function on M such that f is bounded from above and $\Delta f \geq 0$ everywhere on M. Show that there *does not* exist any point $p \in M$ such that $f(p) = \sup\{f(x) : x \in M\}$.

3. Let M be a compact smooth manifold of dimension d. Prove that there exists some $n \in \mathbb{Z}^+$ such that M can be regularly embedded in the Euclidean space $\mathbb{R}^n$.

4. Show that any C^∞ function f on a compact smooth manifold M (without boundary) must have at least two critical points. When M is the 2-torus, show that f must have more than two critical points.

5. Construct a space X with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_2 \times \mathbb{Z}_3$, $H_2(X) = \mathbb{Z}$, and all other homology groups of X vanishing.

6. (a). Define the degree $\deg f$ of a C^∞ map $f : S^2 \longrightarrow S^2$ and prove that $\deg f$ as you present it is well-defined and independent of any choices you need to make in your definition.
(b). Prove in detail that for each integer k (possibly negative), there is a C^∞ map $f : S^2 \longrightarrow S^2$ of degree k.