S.-T. Yau College Student Mathematics Contests 2014

Algebra and Number Theory

Team

Solve 5 out of 6 problems, or the highest 5 scores will be counted.

Problem 1. Let the special linear group (of order 2)

$$\mathrm{SL}_2(\mathbb{R}) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : \det g = 1\}$$

act on the upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ linear fractionally:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

(a) (5 points) Prove that the action is transitive, i.e., for any two $z_1, z_2 \in \mathbb{H}$, there is $g \in \mathrm{SL}_2(\mathbb{R})$ such that $gz_1 = z_2$.

(b) (5 points) For a fixed $z \in \mathbb{H}$, prove that its stabilizer $G_z = \{g \in \mathrm{SL}_2(\mathbb{R}) : gz = z\}$ is isomorphic to $\mathrm{SO}_2(\mathbb{R}) = \{g \in M_2(\mathbb{R}) : gg^t = 1\}$, where g^t is the transpose of g.

(c)(10 points) Let $\mathbb{Z}$ be the set of integers and let

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z}, \quad a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \pmod{2} \right\}$$

be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ (no need to prove this), and let it act on $\mathbb{Q} \cup \{\infty\}$ linearly fractionally as above. How many orbits does this action have? Give a representative for each orbit.

Problem 2. Let $p \geq 7$ be an odd prime number.

(a) (5 points) (to warm up) Evaluate the rational number $\cos(\pi/7) \cdot \cos(2\pi/7) \cdot \cos(3\pi/7)$.

(b) (15 points) Show that $\prod_{n=1}^{(p-1)/2} \cos(n\pi/p)$ is a rational number and determine its value.

Problem 3. (20 points, 10 points each) For any 3×3 matrix $A \in M_3(\mathbb{Q})$, let A^{db} be the 6×6 matrix

$$A^{db} := \begin{pmatrix} 0 & I_3 \\ A & 0 \end{pmatrix}$$

(a) Express the characteristic and minimal polynomials of A^{db} over $\mathbb{Q}$ in terms of the characteristic and minimal polynomial of A.

(b) Suppose that $A, B \in M_3(\mathbb{Q})$ are such that A^{db} and B^{db} are conjugate in the sense that there exists an element $C \in GL_6(\mathbb{Q})$ such that $C \cdot A^{db} \cdot C^{-1} = B^{db}$. Are A and B conjugate? (Either prove this statement or give a counterexample.)

Problem 4. (20 points) Classify all groups of order 8.

Problem 5. Let V be a finite dimensional vector space over complex field $\mathbb{C}$ with a non-degenerate symmetric bilinear form $(\, , \,)$. Let

$$O(V) = \{g \in \mathrm{GL}(V) | (gu, gv) = (u, v), \ u, v \in V\}$$

be the orthogonal group.

(a) (10 points) Prove that

$$(V \otimes_{\mathbb{C}} V)^{O(V)} \cong \mathrm{End}_{O(V)}(V),$$

and construct one such isomorphism. Here $O(V)$ acts on $V \otimes_{\mathbb{C}} V$ via $g(a \otimes b) = ga \otimes gb$, and $(V \otimes_{\mathbb{C}} V)^{O(V)}$ is the fixed point subspace of $V \otimes V$.

(b) (10 points) Prove that the fixed point subspace $(V \otimes_{\mathbb{C}} V)^{O(V)}$ is 1-dimensional.

Problem 6. (20 points) Let c be a non-zero rational integer.

(a) (6 points) Factorize the three variable polynomial

$$f(x, y, z) = x^3 + cy^3 + c^2z^3 - 3cxyz$$

over $\mathbb{C}$ (you may assume $c = \theta^3$ for some $\theta \in \mathbb{C}$).

(b) (7 points) When $c = \theta^3$ is a cube for some rational integer θ, prove that there are only finitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$.

(c) (7 points) When c is not a cube of any rational integers, prove that there infinitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$.