S.-T. Yau College Student Mathematics Contests 2024

Analysis and Differential Equations

Problem 1. Let $Q : \mathbb{R} \to \mathbb{R}$ be a C_c^∞ function, i.e. it is smooth and has compact support. We assume Q is even, i.e. $Q(x) = Q(-x)$. We assume Q is non-trivial, (i.e. Q does not equal to zero everywhere).

Let $T_1(x) := xQ(x)$, and let $T_2(x) = x^2 Q(x)$. Let $T_3 := e^{-x^2}(1 + x^{2024})$
We also introduce the following notation. For any $f : \mathbb{R} \to \mathbb{R}$, $\lambda > 0, \alpha \in \mathbb{R}$, we define

$$f_{\lambda,\alpha}(x) := \frac{1}{\lambda^{1/2}} f(\frac{x-\alpha}{\lambda}) \tag{0.1}$$

We claim: There exists $\delta > 0, \epsilon > 0$, so that for any $c \in \mathbb{R}$ with $|c| < \delta$, one can find unique λ, α such that the followings hold

1. $|\lambda - 1| + |\alpha| < \epsilon$.
2. $< Q_{\lambda,\alpha} - Q - cT_3, \quad T_1 >= 0$
3. $< Q_{\lambda,\alpha} - Q - cT_3, \quad T_2 >= 0$

(Here, for any two functions f_1, f_2, we define$< f_1, f_2 >:= \int f_1(x) f_2(x) dx$).
Is the above claim correct? Prove your conclusion.

Problem 2 Recall for every $f \in L^2(\mathbb{R}^3)$, one has that $g(x) := (-\Delta + 1)^{-1} f$ is a well-defined $L^2(\mathbb{R}^3)$ function. And one may compute g by solving

$$(-\Delta + 1)g = f \tag{0.2}$$

(Recall Δ in $\mathbb{R}^3$ is defined as $\Delta := \sum_{i=1}^3 \partial_i^2$, also recall one may also define $(-\Delta + 1)^{-1}$ by Fourier theory.)

Now, let $V(x) := e^{-|x|^2}$, $x \in \mathbb{R}^3$. Prove that the operator $T := I + (-\Delta + 1)^{-1} V$ is invertible in L^2.

(Here, $Tf := f + (-\Delta + 1)^{-1}(Vf)$.)

Problem 3 Let $\psi(\xi) \in C_c^\infty(\mathbb{R})$ be smooth and has compact support. Let $\psi(\xi) = 0, \forall |\xi| \geqslant 1$. Let $f_1(\xi), f_2(\xi) \in C_c^\infty(\mathbb{R})$, i.e. f_1, f_2 are smooth and have compact support. Let $u_i : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $i = 1, 2$, be defined as

$$\begin{aligned} u_1(x_1, x_2) &:= \int_{\mathbb{R}} \psi(\xi) f_1(\xi) e^{i\xi x_1} e^{i\xi^2 x_2} d\xi, \\ u_2(x_1, x_2) &:= \int_{\mathbb{R}} \psi(\eta - 10) f_2(\eta) e^{i\eta x_1} e^{i\eta^2 x_2} d\eta \end{aligned} \tag{0.3}$$

Prove there exists a constant C, which may depend on ψ, but does not depend on f_1, f_2, so that

$$\|u_1 u_2\|_{L^2(\mathbb{R}^2)} \leqslant C \|f_1\|_{L^2(\mathbb{R})} \|f_2\|_{L^2(\mathbb{R})}. \tag{0.4}$$

(Hint: One may try to use Plancherel Theorem. It may be useful to observe that if one let $H(\xi, \eta) = f_1(\xi) f_2(\eta)$, then $\|H\|_{L^2(\mathbb{R}^2)}$ are also bounded by $\|f_1\|_{L^2(\mathbb{R})} \|f_2\|_{L^2(\mathbb{R})}$)

Problem 4 Consider the heat equation in $\mathbb{R}^2$. Let $u = u(t,x)$ is a solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0; \\ u|_{t=0} = u_0 \in L^2. \end{cases}$$

Then there exists a universal constant C such that

$$\int_0^\infty \|u(t)\|_{L^\infty}^2 dt \leqslant C\|u_0\|_{L^2}^2.$$

Problem 5 . Consider the Fourier transform. Let

$$Q(g,f)(x) := \int_{\mathbb{R}^N} \int_{\mathbf{S}^{N-1}} B(|x-y|, \frac{x-y}{|x-y|} \cdot \sigma) g(y') f(x') d\sigma dy,$$

where B is a given two variable function, $\mathbf{S}^{N-1}$ stands for the unit sphere in $\mathbb{R}^N$ and

$$x' := \frac{x+y}{2} + \frac{|x-y|\sigma}{2}; \quad y' := \frac{x+y}{2} - \frac{|x-y|\sigma}{2}. \tag{0.5}$$

Then

$$\widehat{Q(g,f)}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N \times \mathbf{S}^{N-1}} \hat{B}(|\eta|, \frac{\xi}{|\xi|} \cdot \sigma) \hat{g}(\xi^- + \eta) \hat{f}(\xi^+ - \eta) d\sigma d\eta,$$

where $\hat{B}(|\eta|, t) := \int_{\mathbb{R}^N} B(|q|, t) e^{-iq\cdot\eta} dq$, $\xi^\pm := \frac{\xi \pm |\xi|\sigma}{2}$.