S.-T. Yau College Student Mathematics Contests 2015

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Let $SO(3)$ be the set of all 3×3 real matrices A with determinant 1 and satisfying ${}^tAA = I$, where I is the identity matrix and ${}^tA$ is the transpose of A. Show that $SO(3)$ is a smooth manifold, and find its fundamental group. You need to prove your claims.

2. Let X be a topological space. The *suspension* $S(X)$ of X is the space obtained from $X \times [0,1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.

3. Let $F : M \to N$ be a smooth map between two manifolds. Let X_1, X_2 be smooth vector fields on M and let Y_1, Y_2 be smooth vector fields on N. Prove that if $Y_1 = F_* X_1$ and $Y_2 = F_* X_2$, then $F_*[X_1, X_2] = [Y_1, Y_2]$, where $[\ ,\ ]$ is the Lie bracket.

4. Let M_1 and M_2 be two compact convex closed surfaces in $\mathbb{R}^3$, and $f : M_1 \to M_2$ a diffeomerphism such that M_1 and M_2 have the same inner normal vectors and Gauss curvatures at the corresponding points. Prove that f is a translation.

5. Prove the second Bianchi identity:

$$R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l} = 0$$

6. Let M_1, M_2 be two complete n-dimensional Riemannian manifolds and $\gamma_i : [0,a] \to M_i$ are two arc length parametrized geodesics. Let ρ_i be the distance function to $\gamma_i(0)$ on M_i. Assume that $\gamma_i(a)$ is within the cut locus of $\gamma_i(0)$ and for any $0 \le t \le a$ we have the inequality of sectional curvatures

$$K_1(X_1, \frac{\partial}{\partial \gamma_1}) \ge K_2(X_2, \frac{\partial}{\partial \gamma_2}),$$

where $X_i \in T_{\gamma_i(t)} M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}$.

Then

$$Hess(\rho_1)(\widetilde{X}_1, \widetilde{X}_1) \le Hess(\rho_2)(\widetilde{X}_2, \widetilde{X}_2),$$

where $\widetilde{X}_i \in T_{\gamma_i(a)} M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}(a)$.