S.-T. Yau College Student Mathematics Contests 2017

# Analysis and Differential Equations

## Individual

Please solve 5 out of the following 6 problems.

**1.** Suppose $f$ is integrable on $[-\pi, \pi]$, prove that $\Sigma_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx}$ tends to $f(x)$ for a.e. $x$, as $r \to 1, r < 1$. Here $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$.

**2.** Let $H$ be a Hilbert space equipped with an inner product $(.,.)$ and a norm $||.|| = (.,.)^{\frac{1}{2}}$. A sequence $\{f_k\}$ is converge to $f \in H$ if $||f_k - f|| \to 0$. A sequence $\{f_k\} \subset H$ is said converge weakly to $f \in H$ if $(f_k, g) \to (f, g)$ for any $g \in H$. Prove the following statements:

a) $\{f_k\}$ converges to $f$ if and only if $||f_k|| \to ||f||$ and $\{f_k\}$ converges weakly to $f$.

b) If $H$ is a finite dimensional Hilbert space, then the weak convergence implies convergence. Give a counter example to show that weak convergence does not necessarily imply convergence in an infinite dimensional Hilbert space.

**3.** Let $f : \mathbf{C}/\{0\} \to \mathbf{C}$ be a holomorphic function and

$$|f(z)| \leq |z|^2 + \frac{1}{|z|^{1/2}},$$

for $z$ near 0. Determine all such functions.

**4.** Find a conformal mapping which maps the region $\{z||z - i| < \sqrt{2}, |z + i| < \sqrt{2}\}$ onto the unit disk.

**5.** If $E$ is a compact set in a region $\Omega$, prove that there exists a constant $M > 0$, depending only on $E$ and $\Omega$, such that every positive harmonic function $u(z)$ in $\Omega$ satisfies $u(z_2) \leq Mu(z_1)$ for any two points $z_1, z_2 \in E$.

**6.** 1) For any bounded domain $\Omega \subset \mathbf{R}^n$, there exists a smallest constant $C(\Omega)$, such that

$$\int_\Omega |u|^2 dx \leq C(\Omega) \int_\Omega \Sigma_{i=1}^n |\frac{\partial u}{\partial x_i}|^2 dx$$

for every function $u \in H_0^1(\Omega) = \overline{C_0^\infty(\Omega)} \subset H^1(\Omega)$, where $C_0^\infty(\Omega)$ is the space of smooth functions over $\Omega$ and vanishing on boundary of $\Omega$ and $H^1(\Omega)$ is the Banach space of functions $u \in L^2(\Omega), \nabla u \in L^2(\Omega)^{\otimes n}$ with the norm:

$$||u||_{H^1(\Omega)}^2 = ||u||_{L^2(\Omega)}^2 + ||\nabla u||_{L^2(\Omega)^{\otimes n}}^2$$

$$= \int_\Omega (|u|^2 + \Sigma_{i=1}^n |\frac{\partial u}{\partial x_i}|^2) dx.$$

$H_0^1(\Omega)$ is the completion of $C^\infty(\Omega)$ in $H^1(\Omega)$ with the above norm.

2) Let $\Pi = \{(x, y)|0 < x < a, 0 < y < b\}$, show that $C(\Pi) \geq \frac{a^2 b^2}{\pi^2(a^2+b^2)}$.