S.-T. Yau College Student Mathematics Contests 2019

Probability and Statistics

Team (4 problems)

1) Suppose $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables and the common law is exponential with parameter one. Show that

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{X_n}{\log n} = 1\right] = 1.$$

2) Let $(X_n)_{n\geq 1}$ be i.i.d. real random variables and set $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$. Suppose that for some constant $c \in \mathbb{R}$ we have $S_n/n \to c$ as $n \to \infty$ almost surely. Show that X_1 has a finite first moment and $\mathbb{E}[X_1] = c$.

3) Consider uniform permutation of $\{1, 2, \ldots, n\}$ and denote by X_n the number of cycles in the permutation. Find a sequence of reals $(a_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} \frac{\mathbb{E}[X_n]}{a_n} = 1,$$

and justify your answer.

4) The Erdös-Rényi random graph $G(n, p)$ with parameters $n \geq 1$ and $p \in [0, 1]$ is the random graph whose vertex set is $V = \{1, 2, \ldots, n\}$ and where for each pair $i \neq j \in V$ the edge $i \leftrightarrow j$ is present with probability p independently of all the other pairs.

   (a) For $\epsilon > 0$, if $p_n \geq (1+\epsilon)\frac{\log n}{n}$, then

   $$\mathbb{P}[G(n, p_n) \text{ has an isolated vertex}] \to 0, \quad \text{as } n \to \infty.$$

   (b) For $\epsilon > 0$, if $p_n \leq (1-\epsilon)\frac{\log n}{n}$, then

   $$\mathbb{P}[G(n, p_n) \text{ has an isolated vertex}] \to 1, \quad \text{as } n \to \infty.$$