S.-T. Yau College Student Mathematics Contests 2022
Mathematical Physics

说明：Solve every problem

1 Problems

1. (a) A symmetry transformation in quantum mechanics is represented by a unitary or anti-unitary operator acting on a Hilbert space. The time reversal transformation Θ relates the wave function at time t to time $-t$. Prove: Θ is an anti-unitary operator.

   (5 points) **solution**: We have the evolution operator $U(t,0) = e^{-iHt/\hbar}$, and so we have

   $$|\psi, t\rangle = U(t,0)|\psi, 0\rangle$$

   Here $|\psi, 0\rangle$ is the state at $t = 0$. If a system is invariant under time-reversal symmetry, then its evolution operator satisfies

   $$\Theta^{-1}U(t,0)\Theta = U^{\dagger}(t,0)$$

   Infinitesimally, we have

   $$(1 - itH/\hbar)\Theta = \Theta(1 + itH/\hbar)$$

   and

   $$(-itH/\hbar)\Theta = \Theta(itH/\hbar)$$

   We have: if Θ is a Unitary operator, then $H\Theta = -\Theta H$, this is contradictory as if $|n\rangle$ is an energy state with eigenvalue E_n, then $\Theta|n\rangle$ would be an energy eigenstate with energy $-E_n$, this is contradictory even for the free particle. So Θ would be an anti-Unitary operator.

   (b) Consider state vector $|\psi\rangle$ for a quantum system. A time reversal transformation is represented by an anti-unitary operator Θ. We now consider position space wavefunction $\psi(x) = \langle x|\psi\rangle$, and $\Theta|x\rangle = |x\rangle$. Prove: the position space wave function for $\Theta|\psi\rangle$ is

   $$\psi(x)^*$$

   (5 points) **solution**: The state $|\psi\rangle$ can be expanded using the position eignenstate $|x\rangle$ as follows

   $$|\psi\rangle = \int dx|x\rangle\langle x|\psi\rangle$$

   Then we have (using the anti-unitary property of Θ)

   $$\Theta|\psi\rangle = \Theta(\int dx|x\rangle\langle x|\psi\rangle) = \int dx(\langle x|\psi\rangle)^*\Theta|x\rangle = \int dx(\langle x|\psi\rangle)^*|x\rangle$$

   so $\Theta|\psi\rangle$ has position wave function $\langle x|\psi\rangle)^* = \psi(x)^*$.

(c) A one dimensional quantum system is invariant under time reversal transformation, and so its Hamiltonian satisfies $\Theta H = H\Theta$. If an energy eigenstate $|\psi\rangle$ has no degeneracy, Prove: it is possible to take the position space energy eigenfunction to be real:

$$\psi(x)^* = \psi(x)$$

(5 points) **solution**: For an energy eigenstate $|n\rangle$, so $H|n\rangle = E_n|n\rangle$. Then $\Theta|n\rangle$ is an energy eigenstate with energy E_n. Since E_n has no denegeracy, $|n\rangle$ and $\Theta|n\rangle$ has to be linearly dependent, namely there is a complex number λ such that

$$\Theta|n\rangle = \lambda|n\rangle$$

Since the wave function for $\Theta|n\rangle$ is $\psi(x)^*$, we have the equation

$$\psi^*(x) = \lambda\psi(x)$$

Since the wave function has the freedom of multiplying a complex number with $|\lambda| = 1$, we can use this freedom to choose the wave function to be real.

2. Consider following quantum Hamiltonian:

$$H_0 = \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_2^2$$

This is the Hamiltonian for two decoupled harmonic oscillators.

(a) Calculate the eigenstates and eigenvalues for H_0 (an energy eigenstate could be labeled as $|n_1, n_2\rangle$).

(5 points) **Solution**: Define operators

$$a_1 = \sqrt{\frac{mw}{2\hbar}}x_1 + i\sqrt{\frac{1}{2\hbar wm}}p_1 \quad , a_1^\dagger = \sqrt{\frac{mw}{2\hbar}}x_1 - i\sqrt{\frac{1}{2\hbar wm}}p_1,$$
$$a_2 = \sqrt{\frac{mw}{2\hbar}}x_2 + i\sqrt{\frac{1}{2\hbar wm}}p_2, \quad a_2^\dagger = \sqrt{\frac{mw}{2\hbar}}x_2 - i\sqrt{\frac{1}{2\hbar wm}}p_2$$

they satisfy the nontrivial commutation relation

$$[a_1, a_1^\dagger] = 1, \quad [a_2, a_2^\dagger] = 1$$

The Hamiltonian becomes

$$H_0 = \hbar w(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)$$

The eigenstates are found by starting with a state $|0,0\rangle$ which satisfies the condition

$$a_1|0,0\rangle = 0, \quad a_2|0,0\rangle = 0$$

An energy eigenstate is formed by the state

$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} |0, 0\rangle$$

and the energy is given as

$$(n_1 + n_2 + 1)\hbar\omega$$

(b) Assume the creation and annihilation operators for two harmonic oscillators are $a_i^\dagger, a_i, i = 1, 2$. Define following operators

$$J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$$

i. Prove that: $[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z$.
(5 points) **Solution**: Using the commutation relation

$$[a_1, a_1^\dagger] = 1, \quad [a_2, a_2^\dagger] = 1$$

to directly verify the commutation relation.

ii. Consider one eigenvalue E_n of H_0, (here $n_1 + n_2 = n$). Prove that: all eigenstates of E_n form an irreducible representation of $su(2)$ Lie algebra, and compute the spin.
(5 points) **Solution**: The energy eigenstates of E_n has degeneracy $n + 1$, which form a space M_n on which there is a $su(2)$ lie algebra action, with the operators $J_z, J_\pm$. Consider an energy eigenstate $|n_1, n_2\rangle$, we have

$$J_z|n_1, n_2\rangle = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0, 0\rangle$$
$$= \frac{1}{2}(n_1 - n_2)|n_1, n_2\rangle$$

so $|n_1, n_2\rangle$ is the eigenstate of J_z with eigenvalue $\frac{1}{2}(n_1 - n_2)$. The maximal eigenvalue is $\frac{n}{2}$, and the minimal eigenvalue is $-\frac{1}{2}(n)$. So it forms a spin $\frac{n}{2}$ representation.

(c) Consider following perturbed Hamiltonian (λ is small)

$$H = H_0 + \lambda x_1^2 p_2^2$$

Compute the first order correction to the energy for the energy level $n_1 + n_2 = 2$.
(10 points) **Solution**: There are a total of three states $\alpha_1 = |0, 2\rangle, \alpha_2 = |1, 1\rangle, \alpha_3 = |2, 0\rangle$ for $n = 2$. We need to compute the three by three matrix

$$\langle n_1, n_2 | x_1^2 p_2^2 | n_1', n_2' \rangle$$

and then compute the eigenvalues of this matrix. Since x_1 and p_2 commute, and so

$$\langle n_1, n_2|x_1^2 p_2^2|n_1^{'}, n_2^{'}\rangle = \langle n_1|x_1^2|n_1^{'}\rangle\langle n_2|p_2^2|n_2^{'}\rangle$$

Next, using the expansion in creation of annihilation operators:

$$x_1^2 = \frac{\hbar}{2mw}(a_1^2+(a_1^\dagger)^2+a_1a_1^\dagger+a_1^\dagger a_1),\ \ p_2^2 = \frac{\hbar}{2mw}(a_2^2+(a_2^\dagger)^2-a_2a_2^\dagger-a_2^\dagger a_2)$$

The nonzero matrix element for x_1^2 is $\langle 0|x_1^2|2\rangle, \langle 0|x_1^2|0\rangle, \langle 1|x_1^2|1\rangle, \langle 2|x_1^2|2\rangle$ (and conjugate), and their values are (we ignore the factor $\frac{\hbar}{2mw}$)

$$\langle 0|x_1^2|2\rangle = \sqrt{2},\quad \langle 0|x_1^2|0\rangle = 1,\ \langle 1|x_1^2|1\rangle = 3,\ \langle 2|x_1^2|2\rangle = 5$$

similarly the nonzero matrix element for p_2^2 is $\langle 0|p_2^2|2\rangle, \langle 0|p_2^2|0\rangle, \langle 1|p_2^2|1\rangle, \langle 2|p_2^2|2\rangle$ (and conjugate), and their values are

$$\langle 0|p_2^2|2\rangle = \sqrt{2},\quad \langle 0|p_2^2|0\rangle = -1,\ \langle 1|p_2^2|1\rangle = -3,\ \langle 2|p_2^2|2\rangle = -5$$

So the non-zero matrix element is

$$\langle 0,2|x_1^2p_2^2|2,0\rangle,\quad \langle 0,2|x_1^2p_2^2|0,2\rangle,\ \langle 1,1|x_1^2p_2^2|1,1\rangle,\ \langle 2,0|x_1^2p_2^2|2,0\rangle,$$

and the matrix is given as

$$\begin{bmatrix} -5 & 0 & 2 \\ 0 & -9 & 0 \\ 2 & 0 & -5 \end{bmatrix}$$

The eigenvalue of above matrix is given as $\lambda_1 = -9, \lambda_2 = -7, \lambda_3 = -3$.

3. A Killing vector field $k^\mu \frac{\partial}{\partial x^\mu}$ satisfies the equation $k^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu k^\lambda g_{\lambda\nu} + \partial_\nu k^\lambda g_{\lambda\mu} = 0$.

   (a) Prove: $D_\mu k_\nu + D_\nu k_\mu = 0$, here D_μ is the covariant derivative. (5 points) **Solution**: By definition

$$D_\mu k_\nu + D_\nu k_\mu = \partial_\mu k_\nu - \Gamma^\rho_{\mu\nu} k_\rho + \partial_\nu k_\mu - \Gamma^\rho_{\mu\nu} k_\rho$$

Since the connection is given as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

The above equation simplifies

$$\begin{aligned} & D_\mu k_\nu + D_\nu k_\mu = \partial_\mu k_\nu + \partial_\nu k_\mu - k_\rho g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \\ & = \partial_\mu k_\nu + \partial_\nu k_\mu - k^\sigma(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \\ & = \partial_\mu(k^\sigma g_{\nu\sigma}) + \partial_\nu(k^\sigma g_{\mu\sigma}) - k^\sigma(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \\ & = g_{\nu\sigma}\partial_\mu k^\sigma + g_{\mu\sigma}\partial_\nu k^\sigma + k^\sigma \partial_\sigma g_{\mu\nu} \\ & = 0 \end{aligned}$$

In the last line, we use the definition of Killing vector field.

(b) For a moving particle in gravitational background with a Killing vector field, Prove: $k^\mu P_\mu$ is a conserved quantity, Here $P_\mu = m\frac{dx^\nu}{d\tau} g_{\mu\nu}$ is the momentum for the free falling particle with trajectory $x^\nu(\tau)$.

(10 points) **Solution**: We need to verify

$$\frac{d}{d\tau}(k^\mu P_\mu) = 0$$

Substitute the definition of P_μ, we need to compute

$$\begin{aligned}&\frac{d}{d\tau}(k^\mu \dot{x}^\nu g_{\mu\nu}) = \frac{d}{d\tau}(k_\mu \dot{x}^\mu) = \dot{x}^\rho \partial_\rho (k_\mu \dot{x}^\mu)\\ &= \dot{x}^\rho \dot{x}^\mu D_\rho(k_\mu) + \dot{x}^\rho k_\mu D_\rho \dot{x}^\mu\end{aligned}$$

The first term is vanishing due to the fact $D_\rho k_\mu = -D_\mu k_\rho$, which is valid because k_μ is the Killing vector field. The second term vanishes by using the equation of motion for the free falling particle

$$\begin{aligned}&\frac{d}{d\tau}\dot{x}^\mu - \Gamma^\mu_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma = 0 \to\\ &\dot{x}^\rho D_\rho \dot{x}^\mu = 0\end{aligned}$$

4. Consider following metric

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + drdv + r^2 d\Omega^2$$

Here $d\Omega^2$ is the standard metric on two sphere. Consider the hypersurface defined by $S = r - 2M = 0$, and a vector field $l = \tilde{f}(x)(g^{\mu\nu}\partial_\nu S)\frac{\partial}{\partial x^\mu}$, here $\tilde{f}(x)$ is a non-zero function. Prove:

(a) l is normal to the surface S.

(5 points) **Solution**: In the particular metric, $l = \tilde{f}(x)\frac{\partial}{\partial v}$ The tangent space for S is generated by the vector $(\partial_v, \partial_\theta, \partial_\phi)$. We have the inner product

$$g(\tilde{f}(x)\frac{\partial}{\partial v}, \partial_v) \propto g_{vv} = (1 - \frac{2M}{r}),\ g(\tilde{f}(x)\frac{\partial}{\partial v}, \partial_v\theta) = 0,\ g(\tilde{f}(x)\frac{\partial}{\partial v}, \partial_\phi) = 0$$

On S, we have $r = 2M$, so l is normal to the tangent space of S.

(b) $l^2 = 0$ on the surface S.

(5 points) **Solution**: Direct computation:

$$l^2 = g(\tilde{f}(x)\frac{\partial}{\partial v}, \tilde{f}(x)\frac{\partial}{\partial v}) = \tilde{f}(x)^2 g_{vv} = \tilde{f}(x)^2(1 - \frac{2M}{r})$$

on S, we have $r = 2M$, and so $l^2 = 0$.

(c) $\frac{\partial}{\partial v}$ is a Killing vector field.
(5 points) **Solution**: Using the definition of Killing vector field $k^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu k^\lambda g_{\lambda\nu} + \partial_\nu k^\lambda g_{\lambda\mu} = 0$. For the vector field $\frac{\partial}{\partial v}$, only the component $k^v = 1$ which is constant, other components are zero, so the Killing equation becomes

$$k^\lambda \partial_\lambda g_{\mu\nu} = k^v \partial_v (g_{\mu\nu})$$

Since the coefficient of the metric do not depend on v, the above equation is zero.

5. The energy momentum tensor for a relativistic quantum field theory is denoted as $\theta^{\mu\nu}$, which is symmetric and conserved.

(a) Define new current $s^\mu = x_\nu \theta^{\mu\nu}$ and $K^{\lambda\mu} = x^2\theta^{\lambda\mu} - 2x^\lambda x_\rho \theta^{\rho\mu}$. Compute $\partial^\mu s_\mu$ and $\partial_\mu K^{\lambda\mu}$, and explain the condition on $\theta^{\mu\nu}$ so that these new currents are conserved.
(5 points) **solution**: Direct computation

$$\partial^\mu s_\mu = \theta^\mu_\mu, \quad \partial_\mu K^{\lambda\mu} = -2x^\lambda \theta^\mu_\mu$$

These new currents are conserved if θ is traceless $\theta^\mu_\mu = 0$.

(b) Consider a scalar field $\sigma(x)$ which transforms under a scale transformation as

$$\delta\sigma = x^\lambda \partial_\lambda \sigma + f^{-1}$$

we have following Lagrangian

$$L = L_s - \frac{\mu_0^2}{2}\phi^2 e^{2f\sigma} + \frac{1}{2f^2}\partial_\mu e^{f\sigma}\partial^\mu e^{f\sigma}$$

The infinitesimal scale transformation on scalar field ϕ is $\delta\phi = (1 + x_\lambda \partial^\lambda)\phi$. Here L_s is scale invariant part of the Lagrangian. Prove that: the above Lagrangian is scale invariant.
(10 points) **solution**: We have

$$\delta L = \delta L_s - \mu_0^2 \delta\phi\phi e^{2f\sigma} - \frac{\mu_0^2}{2}\phi^2 e^{2f\sigma} 2f\delta\sigma + \frac{1}{f^2}\partial_\mu[e^{f\sigma} f\delta\sigma]\partial^\mu e^{f\sigma}$$

substitute

$$\delta\phi = (1 + x_\lambda \partial_\lambda)\phi, \quad \delta\sigma = x^\lambda \partial_\lambda \sigma + f^{-1}$$

and using $\delta L_s = 0$, we have

$$\begin{aligned}
\delta L &= -\mu_0^2 e^{2f\sigma}\phi(1 + x^\lambda\partial_\lambda)\phi - (\mu_0^2)\phi^2 e^{2f\sigma} f(x^\lambda \partial_\lambda \sigma + f^{-1}) \\
&+ \frac{1}{f^2}\partial_\mu[e^{f\sigma} f(x^\lambda\partial_\lambda\sigma + f^{-1})]\partial^\mu e^{f\sigma} \\
&= (4 + x^\lambda \partial_\lambda)(-\frac{1}{2}\mu_0^2\phi^2 e^{2f\sigma} + \frac{1}{2f^2}\partial_\mu e^{f\sigma}\partial^\mu e^{f\sigma}]
\end{aligned}$$

Here we assume the theory is a 4d theory, and so by integrating by parts, the above Lagrangian is scale invariant.

(c) Explain why a classically scale invariant Lagrangian for a quantum field theory may fail to be scale invariant quantum mechanically.

(5 points) **solution**: For the perturbative quantum field theory, to deal with divergence of loop diagrams, one need to do regularization and renormalization. In doing regularization, we introduce a scale which could spoil the classical scale invariance. This happens for four dimensional $\lambda\phi^4$ theory. It might be possible that one can find a regularization and renormalization scheme such that scale invariance is preserved quantum mechanically, this happens for four dimensional $\mathcal{N} = 4$ supersymmetric field theory.

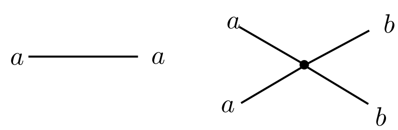
6. Consider following Lagrangian for N scalar fields $\phi^a, a = 1, \ldots, N$:

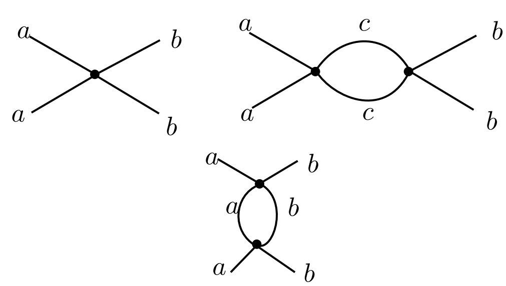
$$L = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}\mu_0^2 \phi^a \phi^a - \frac{1}{8}\lambda_0 (\phi^a \phi^a)^2$$

Here the repeated index implies the summation over the index.

(a) Write down the propagator and interaction vertex for this model, and write down four point Feynman diagrams up to one loop level.

(5 points) **solution**: See figure.

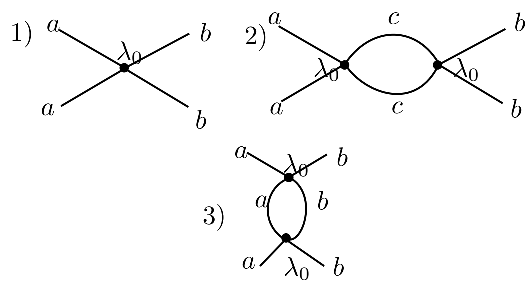




(b) Define $g_0 = \lambda_0 N$, and compute the order in g_0 and N for all the diagrams listed in last question. If we fix the coupling g_0, and let N go to infinity, list the leading order Feynman diagrams in $\frac{1}{N}$.

(5 points) **solution**:

The first diagram has order $\lambda_0 = \frac{g_0}{N}$, and the second diagram has order $(\lambda_0)^2 \times N$, notice that there is an extra factor of N due to the summation of internal scalar of type c, and so the order is $\frac{g_0^2}{N^2} * N = \frac{g_0^2}{N}$. The third diagram has order $\lambda_0^2 = \frac{g_0^2}{N^2}$, and notice that here the type of internal scalar is fixed, so there is no summation.



By fixing g_0, and in the large N limit, the first and second diagram is of $\frac{1}{N}$ order.