S.-T. Yau College Student Mathematics Contests 2011

Geometry and Topology

Team
9:00–12:00 am, July 9, 2011
(Please select 5 problems to solve)

1. Suppose K is a finite connected simplicial complex. True or false:
a) If $\pi_1(K)$ is finite, then the universal cover of K is compact.
b) If the universal cover of K is compact then $\pi_1(K)$ is finite.

2. Compute all homology groups of the the m-skeleton of an n-simplex, $0 \leq m \leq n$.

3. Let M be an n-dimensional compact oriented Riemannian manifold with boundary and X a smooth vector field on M. If $\mathbf{n}$ is the inward unit normal vector of the boundary, show that

$$\int_M div(X)\, dV_M = \int_{\partial M} X \cdot \mathbf{n}\, dV_{\partial M}.$$

4. Let $\mathcal{F}^k(M)$ be the space of all C^∞ k-forms on a differentiable manifold M. Suppose U and V are open subsets of M.
a) Explain carefully how the usual exact sequence

$$0 \longrightarrow \mathcal{F}(U \cup V) \longrightarrow \mathcal{F}(U) \oplus \mathcal{F}V) \longrightarrow \mathcal{F}(U \cap V) \longrightarrow 0$$

arises.
b) Write down the "long exact sequence" in de Rham cohomology associated to the short exact sequence in part (a) and describe explicitly how the map

$$H^k_{deR}(U \cap V) \longrightarrow H^{k+1}_{deR}(U \cup V)$$

arises.

5. Let M be a Riemannian n-manifold. Show that the scalar curvature $R(p)$ at $p \in M$ is given by

$$R(p) = \frac{1}{vol(S^{n-1})} \int_{S^{n-1}} Ric_p(x) dS^{n-1},$$

where $Ric_p(x)$ is the Ricci curvature in direction $x \in S^{n-1} \subset T_pM$, $vol(S^{n-1})$ is the volume of S^{n-1} and dS^{n-1} is the volume element of S^{n-1}.

6. Prove the Schur's Lemma: If on a Riemannian manifold of dimension at least three, the Ricci curvature depends only on the base point but not on the tangent direction, then the Ricci curvature must be constant everywhere, i.e., the manifold is Einstein.