S.-T. Yau College Student Mathematics Contests 2018

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let M and N be smooth, connected, orientable n-manifolds for $n \geq 3$, and let $M\#N$ denote their connect sum.

(a) Compute the fundamental group of $M\#N$ in terms of that of M and of N (you may assume that the basepoint is on the boundary sphere along which we glue M and N).

(b) Compute the homology groups of $M\#N$.

(c) For part (a), what changes if $n = 2$? Use this to describe the fundamental groups of orientable surfaces.

2. Determine all of the possible degrees of maps $S^2 \to S^1 \times S^1$.

3. Classify all vector bundles over the circle S^1 up to isomorphism.

4. Suppose C is a regular curve in the unit sphere S^2. For any point $W \in S^2$, there exists the only oriented great circle S_W (determined by the right hand rule) in S^2 such that W is the pole of S_W. Denote by $n(W)$ the number of points at which the oriented great circle S_W and C intersect. Prove the Crofton formula

$$\iint_{S^2} n(W)dW = 4L,$$

where dW and L is the area element of S^2 and the length of C, respectively.

5. Let M be an n-dimensional closed submanifold in the Euclidean space $\mathbb{R}^{n+p}$. Prove the following inequality

$$\int_M H^n dV \geq vol(S^n),$$

where H and dV is the mean curvature (i.e., norm of the mean curvature vector) and the volume element of M, and S^n is the standard unit sphere of dimension n.

6. Let M be an even dimensional compact and oriented Riemannian manifold with positive sectional curvature. Show that M is simply connected.