S.-T. Yau College Student Mathematics Contests 2011

Analysis and Differential Equations

Team

9:00–12:00 am, July 9, 2011

(Please select 5 problems to solve)

1. Let $H^2(\Delta)$ be the space of holomorphic functions in the unit disk $\Delta = \{|z| < 1\}$ such that $\int_\Delta |f|^2 |dz|^2 < \infty$. Prove that $H^2(\Delta)$ is a Hilbert space and that for any $r < 1$, the map $T : H^2(\Delta) \to H^2(\Delta)$ given by $Tf(z) := f(rz)$ is a compact operator.

2. For any continuous function $f(z)$ of period 1, show that the equation

$$\frac{d\varphi}{dt} = 2\pi\varphi + f(t)$$

has a unique solution of period 1.

3. Let $h(x)$ be a C^∞ function on the real line $\mathbb{R}$. Find a C^∞ function $u(x, y)$ on an open subset of $\mathbb{R}^2$ containing the x-axis such that $u_x + 2u_y = u^2$ and $u(x, 0) = h(x)$.

4. Let $S = \{x \in \mathbb{R} \mid |x - \frac{p}{q}| \leq c/q^3,$ for infinitely many relatively prime $p, q \in \mathbb{Z}, q > 0,$ for a fixed $c > 1\}$, show that S is uncountable and its measure is zero.

5. Let $sl(n)$ denote the set of all $n \times n$ real matrices with trace equal to zero and let $SL(n)$ be the set of all $n \times n$ real matrices with determinant equal to one. Let $\varphi(z)$ be a real analytic function defined in a neighborhood of $z = 0$ of the complex plane $\mathbb{C}$ satisfying the conditions $\varphi(0) = 1$ and $\varphi'(0) = 1$.

(a) If φ maps any near zero matrix in $sl(n)$ into $SL(n)$ for some $n \geq 3$, show that $\varphi(z) = \exp(z)$.

(b)Is the conclusion of (a) still true in the case $n = 2$? If it is true, prove it. If not, give a counterexample.

6. Use mathematical analysis to show that:

(a) e and π are irrational numbers;

(b) e and π are also transcendental numbers.