S.-T. Yau College Student Mathematics Contests 2022

Analysis and Differential Equations

Solve every problem.

Problem 1. For $n \geq 1$, we consider the integral

$$I_n = \int_{[0,1]^n} \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \, dx_1 \cdots dx_n.$$

Prove that $\lim_{n\to\infty} I_n$ exists.

Solution: For all positive integers m and n, for all $x, y > 0$, we check that

$$\frac{(m+n)^2}{x+y} \leq \frac{m^2}{x} + \frac{n^2}{y}.$$

Thus, $J_n = nI_n$ satisfies

$$J_{m+n} \leq J_m + J_n.$$

It is well-known that $\lim_{n\to\infty} \frac{J_n}{n}$ exists.

Problem 2. Let $U \subset \mathbf{C}$ be a non-empty open set and $f : U \to U$ be a non-constant holomorphic function. Prove that, if $f \circ f = f$, then $f(z) \equiv z$ for all $z \in U$.

Solution: Since f is not a constant map, $V := f(U) \subset U$ is an nonempty open set (the open mapping property). Thus, for $z \in V$, we have $z = f(z)$. This implies that $f(z) \equiv z$.

Problem 3. Let $X \subset \mathbf{R}$ be a set with positive (Lebesgue) measure. Show that we can find an arithmetic progression of 2022 terms in X, *i.e.*, there exists $x_1, \ldots, x_{2022} \in X$ so that the $x_{i+1} - x_i$'s are all equal and positive, $i = 1, \ldots, 2021$.

Solution: We use m to denote the Lebesgue measure. Let $x \in X$ be a Lebesgue point; therefore, there exists an interval I so that $x \in I$ and $\frac{m(I \cap X)}{m(I)} \geq 1 - \epsilon$ and ϵ will be determined at the end of the proof. By translating and rescaling, we may assume that $I = [0, 1]$. We divide I into 2022 intervals:

$$I = I_1 \cup I_2 \cup \cdots \cup I_{2022}, \;\; I_k = \left[\tfrac{k-1}{2022}, \tfrac{k}{2022}\right], \;\; k = 1, 2, \ldots, 2022.$$

Let $X_k = (I_k \cap X) - \frac{k-1}{2022}$ be the translation of $I_k \cap X$ and $X_k \subset I_1$, $k = 1, \ldots, 2022$. We know that

$$\sum_{k=1}^{2022} m(X_k) \geq 1 - \epsilon.$$

Thus,

$$m\left(\bigcap_{1\leq k\leq 2022} X_k\right) \geq \frac{1}{2022} - 2022\epsilon.$$

We may take $\epsilon = 2023$, thus, $\bigcap_{1\leq k\leq 2022} X_k \neq \varnothing$. Let $x_1 \in \bigcap_{1\leq k\leq 2022} X_k$. Then $x_k = x_1 + \frac{k-1}{2022}$ is the arithmetic progression.

Problem 4. Let $C([0,1])$ be the space of all continuous $\mathbf{C}$-valued functions equipped with L^∞-norm. Let $\mathbf{P} \subset C([0,1])$ be a closed linear subspace. Assume that the elements of $\mathbf{P}$ are polynomials. Prove that $\dim \mathbf{P} < \infty$.

Solution: Let $I = \left\{(x,y) \in [0,1]^2 \;\middle|\; x \neq y\right\}$. For each $(x,y) \in I$, we define a mapping

$$T_{(x,y)} : \mathbf{P} \to \mathbf{C}, \quad u \mapsto \frac{u(x) - u(y)}{|x-y|}.$$

Therefore, we have

$$\sup_{(x,y)\in I} \left|T_{(x,y)}u\right| \leq \|u'\|_{L^\infty}.$$

Since $\mathbf{P}$ is closed, we can apply the Banach-Steinhaus Theorem: there exists $C > 0$, so that

$$\sup_{(x,y)\in I} \left\|T_{(x,y)}\right\|_{\mathbf{P}\to\mathbf{C}} \leq C.$$

We consider the unit ball of $\mathbf{P}$:

$$B = \left\{u \in \mathbf{P} \;\middle|\; \|u\|_{L^\infty} \leq 1\right\}.$$

Hence, for all $u \in B$, we have

$$|u(x) - u(y)| \leq C|x-y|.$$

Thus, the family B is equicontinuous. By the Arzelà-Ascoli Theorem, it is compact. Thus, $\mathbf{P}$ is finite-dimensional.

Problem 5. Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary. Assume that $u \in C\left(\overline{\mathbf{R}^3 - \Omega}\right)$ is a harmonic function on $\mathbf{R}^3 - \Omega$ so that $u\big|_{\Omega} = 1$ and $\lim\limits_{|x|\to\infty} |u(x)| = 0$.

Prove that for such u, $\lim\limits_{|x|\to\infty} |x|u(x)$ exists.

Solution: Let $\varphi(x) \in C^\infty(\mathbf{R}^3)$ so that $\varphi \equiv 0$ on an open neighborhood of Ω and $\varphi \equiv 1$ for $|x| \geq R$ where $R > 0$ is a sufficiently large number. Therefore, we can regard $\varphi \cdot u$ as a smooth function defined on $\mathbf{R}^3$. Hence,

$$\Delta(\varphi u) = \rho,$$

where $\rho \equiv 0$ for $|x| \geq R$. Therefore, for sufficiently large $|x|$, we have

$$\begin{aligned} u(x) &= \varphi(x)u(x) \\ &= -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{\rho(y)}{|y-x|}\,dy \\ &= -\frac{1}{4\pi}\int_{|y|\leq R} \frac{\rho(y)}{|y-x|}\,dy \end{aligned}$$

Therefore,

$$|x|u(x) = -\frac{1}{4\pi}\int_{|y|\leq R} \frac{|x|}{|y-x|}\rho(y)\,dy.$$

Since $|y| \leq R$, $\frac{|x|}{|y-x|}$ converges uniformly to 1 as $|x| \to \infty$, the conclusion follows.

Problem 6. Let $f(x,y) \in C^1(\mathbf{R}^2)$. We assume that there exists $C > 0$ so that for all $(x,y) \in \mathbf{R}^2$, $\left|\frac{\partial f}{\partial y}(x,y)\right| \leq C$. Prove that the following ODE has a globally defined solution for all $y(0) = y_0 \in \mathbf{R}$:

$$\begin{cases} \frac{d}{dx}y(x) = f(x, y(x)), \\ y(0) = y_0. \end{cases} \tag{1}$$

In addition, we assume that f is 1-periodic in x, *i.e.*, for all $(x,y) \in \mathbf{R}^2$, we have $f(x+1,y) = f(x,y)$. Prove that if (1) admits a globally defined bounded solution, then (1) admits a periodic solution.

Solution: The global existence is easy: fix an interval $[0, a)$, we have

$$|y'| \leq |f(x, y(x)) - f(x, y(0))| + |f(x, y(0))| \leq C|y(x) - y(0)| + M \leq C|y| + M.$$

where $M = \sup_{x \in [0,a]} |f(x, y(0))|$. By Gronwall's inequality, y is bounded all the way up to $[0, a]$. We can then extend f across a. This shows the solution can be defined globally.

Assume that φ is a bounded solution. We may assume that $\varphi(1) \neq \varphi(0)$. Otherwise, φ is a periodic solution. Without loss of generality, we may assume that $\varphi(1) > \varphi(0)$. By comparing two solutions $\varphi(x)$ and $\varphi(x+1)$ of (1), we see that $\varphi(0) < \varphi(1) < \cdots < \varphi(n) < \cdots$. Thus, by the boundedness of φ, we may assume that

$$\varphi(n) \to y_* \in \mathbf{R}, \quad n \to \infty.$$

Therefore, the solution to (1) with y_* as the initial data is a 1-periodic solution.