S.-T. Yau College Student Mathematics Contests 2020

Computational and Applied Mathematics

Solve every problem.

Problem 1. Let $f \in C^{k+1}[-1,1]$ and $[-1,1]$ be partitioned into subintervals $I_j = [(j-1)h, jh]$ of width h. Assume p is a polynomial of degree k which approximates f in I_j with

$$\max_{x \in I_j} \left| p_j(x) - f(x) \right| \leq C_0 h^{k+1},$$

where C_0 is a constant independent of j. Show that there exists an another constant C, independent of j, such that

$$\max_{x \in I_{j\pm1}} \left| p_j(x) - f(x) \right| \leq C h^{k+1}.$$

(as long as $I_{j\pm1} \subset [-1,1]$, of course).

Solution: Pick points $0 \leq x_0 < x_1 < \cdots < x_k \leq 1$, and let

$$L_i(x) = \prod_{l \neq i} \frac{x - x_l}{x_i - x_l}$$

be the l-th Lagrange polynomial. Let

$$\Lambda = \sum_{i=0}^{k} \max_{x \in [-1,2]} |L_i(x)|.$$

On I_j we use rescaled versions with $x_{ji} = (j-1)h + hx_i$, and

$$L_{ji}(x) = \prod_{l \neq i} \frac{x - x_{jl}}{x_{ji} - x_{jl}}.$$

Note that Λ is unchanged with

$$\Lambda = \sum_{i} \max_{x \in L_{j\pm1}} \left| L_{ji}(x) \right|.$$

Let f_j be the interpolating polynomial on I_j

$$f_j(x) = \sum_{i=0}^{k} f(x_{ji}) L_{ji}(x),$$

and note that also

$$p_j(x) = \sum_{i} p(x_{ji}) L_{ji}(x).$$

Then for $x \in I_{j\pm1}$,

$$\begin{aligned}
\left| p_j(x) - f(x) \right| &\leq \left| p_j(x) - f_j(x) \right| + \left| f_j(x) - f(x) \right| \\
&= \left| \sum_{i=0}^{k} (f(x_{ji}) - p_j(x_{ji}) L_{ji}(x) \right| + |R_k f(x)| \\
&\leq \max_{x \in I_j} \left| f(x) - p_j(x) \right| \Lambda + \frac{\| f^{(k+1)} \|}{(k+1)!} \max_{x \in I_{j\pm1}} \left| \prod_{i=0}^{k} (x - x_{ji}) \right| \\
&\leq C_0 \Lambda h^{k+1} + \frac{\| f^{(k+1)} \|}{(k+1)!} \max_{x} \left| (2h)^{k+1} \right| \\
&= C h^{k+1},
\end{aligned}$$

$$C = C_0\Lambda + \frac{2^{k+1}\|f^{(k+1)}\|}{(k+1)!}.$$

Problem 2. Consider the iteration

$$x_{n+1} = x_n - \left(\frac{x_n - x_0}{f(x_n) - f(x_0)}\right) f(x_n)$$

for finding the roots of a two times continuous differentiable function $f(x)$. Assuming the method converges to a simple root x^*, what is the rate of convergence? Justify your answer.

Solution: The iteration may be rewritten as

$$x_{n+1} = \frac{[x_n f(x_n) - x_n f(x_0)] - [x_n f(x_n) - x_0 f(x_n)]}{f(x_n) - f(x_0)} = \frac{x_0 f(x_n) - x_n f(x_0)}{f(x_n) - f(x_0)}.$$

Thus

$$x_{n+1} - x^* = \frac{x_0 f(x_n) - x_n f(x_0)}{f(x_n) - f(x_0)} - x^* = \frac{(x_0 - x^*)f(x_n) - (x_n - x^*)f(x_0)}{f(x_n) - f(x_0)}.$$

Taylor's Theorem asserts that there is ξ_n between x_n and x^* such that

$$0 = f(x^*) = f(x_n) + f'(\xi_n)(x^* - x_n) \quad \Rightarrow \quad f(x_n) = f'(\xi_n)(x_n - x^*).$$

This implies

$$x_{n+1} - x^* = \frac{(x_0 - x^*)f'(\xi_n) - f(x_0)}{f(x_n) - f(x_0)}(x_n - x^*).$$

Evaluating the limit as $n \to \infty$, $\xi_n \to x^*$ and

$$\lim_{n\to\infty}\left|\frac{x_{n+1} - x^*}{x_n - x^*}\right| = \lim_{n\to\infty}\left|\frac{(x_0 - x^*)f'(\xi_n) - f(x_0)}{f(x^*) - f(x_0)}\right| = \left|\frac{(x_0 - x^*)\lim_{n\to\infty} f'(\xi_n) - f(x_0)}{0 - f(x_0)}\right|.$$

Applying Taylor's expression one more time, we know there is η between x^* and x_0 such that

$$f(x_0) = f(x^*) + f'(x^*)(x_0 - x^*) + \frac{f''(\eta)}{2}(x_0 - x^*)^2,$$

So

$$f'(x^*)(x_0 - x^*) - f(x_0) = -\frac{f''(\eta)}{2}(x_0 - x^*)^2.$$

Therefore

$$\lim_{n\to\infty}\left|\frac{x_{n+1} - x^*}{x_n - x^*}\right| = \left|\frac{f''(\eta)}{2f(x_0)}\right|(x_0 - x^*)^2.$$

Note the right hand side is dependent only upon x^* and x_0. Since we know $x_n \to x^*$, this shows the rate of convergence is linear.

Problem 3. Suppose $\mathbf{A}$ is an $m \times m$ matrix with a complete set of orthonormal eigenvectors $\mathbf{q_1}, \ldots, \mathbf{q_m}$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_m$. Assume that $|\lambda_1| > |\lambda_2| > |\lambda_3|$ and $\lambda_j \geq \lambda_{j+1}$ for $j = 3, \ldots, m$. Consider the power method $\mathbf{v}^{(k)} = \mathbf{A}\mathbf{v}^{(k-1)}/\lambda_1$, with $\mathbf{v}^{(0)} = \alpha_1\mathbf{q}_1 + \cdots + \alpha_m\mathbf{q}_m$ where α_1 and α_2 are both nonzero. Show that the sequence $\{\mathbf{v}^{(k)}\}_{k=0}^{\infty}$ converges linearly to $\alpha_1\mathbf{q}_1$ with asymptotic constant $C = |\lambda_2/\lambda_1|$.

Solution: Matrix $\mathbf{A}$ has following eigen-decomposition

$$\mathbf{A} = \begin{bmatrix}\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix} \begin{bmatrix}\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\end{bmatrix}^{-1},$$

thus

$$\mathbf{A}^k = \left[\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\right] \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_m^k \end{bmatrix} \left[\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\right]^{-1}.$$

The power method reduces to

$$\begin{aligned}
\mathbf{v}^{(k)} &= \mathbf{A}^k \frac{\mathbf{v}^{(0)}}{\lambda_1^k} \\
&= \left[\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\right] \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_m^k \end{bmatrix} \left[\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\right]^{-1} \left[\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\right] \begin{bmatrix} \frac{\alpha_1}{\lambda_1^k} \\ \frac{\alpha_2}{\lambda_1^k} \\ \vdots \\ \frac{\alpha_m}{\lambda_1^k}, \end{bmatrix} \\
&= \alpha_1 \mathbf{q}_1 + \sum_{j=2}^{m} \left(\frac{\lambda_j}{\lambda_1}\right)^k \alpha_j \mathbf{q}_j,
\end{aligned}$$

from this we deduce $\mathbf{v}^{(k)} \to \alpha_1 \mathbf{q}_1$ as $k \to \infty$, since $|\lambda_j/\lambda_1| < 1$ for $j = 2, \ldots, m$.

To show the convergencee is linear with asymptotic constant $C = |\lambda_2/\lambda_1|$ we need to verify the limit

$$\lim_{k\to\infty} \frac{||e^{(k+1)}||}{||e^{(k)}||} = \lim_{k\to\infty} \frac{||\mathbf{v}^{(k+1)} - \alpha_1 \mathbf{q}_1||}{||\mathbf{v}^{(k)} - \alpha_1 \mathbf{q}_1||} = \left|\frac{\lambda_2}{\lambda_1}\right| \qquad \text{(here } ||\cdot|| \text{ denotes the } L_2\text{-norm).}$$

Note that $e^{(k)} = \sum_{j=2}^{m} \left(\frac{\lambda_j}{\lambda_1}\right)^k \alpha_j \mathbf{q}_j$, using the orthonormality of the eigenvectors we have

$$||e^{(k)}||^2 = \sum_{j=2}^{m} \left|\frac{\lambda_j}{\lambda_1}\right|^{2k} |\alpha_j|^2 = \left|\frac{\lambda_2}{\lambda_1}\right|^{2k} \left(|\alpha_2|^2 + \sum_{j=3}^{m} \left|\frac{\lambda_j}{\lambda_2}\right|^{2k} |\alpha_j|^2\right),$$

similarly

$$||e^{(k+1)}||^2 = \left|\frac{\lambda_2}{\lambda_1}\right|^{2(k+1)} \left(|\alpha_2|^2 + \sum_{j=3}^{m} \left|\frac{\lambda_j}{\lambda_2}\right|^{2(k+1)} |\alpha_j|^2\right).$$

Thus

$$\begin{aligned}
\lim_{k\to\infty} \frac{||e^{(k+1)}||}{||e^{(k)}||} &= \lim_{k\to\infty} \left(\frac{\left|\frac{\lambda_2}{\lambda_1}\right|^{2(k+1)} \left(|\alpha_2|^2 + \sum_{j=3}^{m} \left|\frac{\lambda_j}{\lambda_2}\right|^{2(k+1)} |\alpha_j|^2\right)}{\left|\frac{\lambda_2}{\lambda_1}\right|^{2k} \left(|\alpha_2|^2 + \sum_{j=3}^{m} \left|\frac{\lambda_j}{\lambda_2}\right|^{2k} |\alpha_j|^2\right)} \right)^{\frac{1}{2}} \\
&= \left|\frac{\lambda_2}{\lambda_1}\right| \frac{|\alpha_2|}{|\alpha_2|} \qquad \text{(we have used } |\lambda_2| > |\lambda_3| \geq |\lambda_j| \text{ for } j > 3 \text{)} \\
&= \left|\frac{\lambda_2}{\lambda_1}\right| \qquad \text{(since } \alpha_2 \neq 0 \text{).}
\end{aligned}$$

Problem 4. For the initial value problem $y' = f(t, y)$, $y(0) = y_0$ on the interval $[0,\ T]$, consider the implicit two-step method

$$\begin{aligned}
y_{n+1} &= \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2h}{3} f(t_{n+1}, y_{n+1}), \\
y_1 &= y_0 + h f(t_1, y_0),
\end{aligned}$$

where h is the step size and $t_n = nh$.

(a) What is the order of the accuracy of the scheme?

(b) Check the stability of the scheme by analyzing the stability polynomial?

(c) Find the stability region of the scheme.

Solution: **(a)** Let $y(t)$ be the exact solution, then the truncation error of form

$$h\tau_{n+1} := y(t_{n+1}) - \left(\frac{4}{3}y(t_n) - \frac{1}{3}y(t_{n-1}) + \frac{2h}{3}f(t_{n+1}, y(t_{n+1}))\right)$$

can be estimated by using Taylor expansion to every term involved:

$$\begin{aligned}
y(t_{n+1}) &= y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{1}{6}h^3y_n''' + O(h^4),\\
-\frac{1}{3}y_{n-1} &= -\frac{1}{3}y_n + \frac{1}{3}hy_n' - \frac{1}{6}h^2y_n'' + \frac{1}{18}h^3y_n''' + O(h^4),\\
\frac{2h}{3}f(t_{n+1}, y_{n+1}) &= \frac{2h}{3}y_{n+1}' = \frac{2}{3}hy_n' + \frac{2}{3}h^2y_n'' + \frac{1}{3}h^3y_n''' + O(h^4).
\end{aligned}$$

Hence

$$\begin{aligned}
h\tau_{n+1} &= \left[y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{1}{6}h^3y_n''' + O(h^4)\right] - \left[y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{7}{18}h^3y_n''' + O(h^4)\right]\\
&= -\frac{2}{9}h^3y_n''' + O(h^4).
\end{aligned}$$

The method has order of accuracy 2.

(b) Apply the method to the case $f = 0$, then

$$y_{n+1} - \frac{4}{3}y_n + \frac{1}{3}y_{n-1} = 0,$$

when for ansatz of form $y_n = \gamma^n$ gives the stability polynomial

$$\gamma^2 - \frac{4}{3}\gamma + \frac{1}{3} = 0,$$

which has nonzero roots $\gamma = 1, \frac{1}{3}$. Since $|\gamma| \le 1$ and $\gamma = 1$ as a single root, the method is stable.

(c) Consider the stiff problem $y' = \lambda y$. The method becomes

$$y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1} + \frac{2}{3}h\lambda y_{n+1},$$

which has stability polynomial

$$(3 - 2h\lambda)\gamma^2 - 4\gamma + 1 = 0.$$

So the stability region is given by

$$\left|\frac{4 \pm \sqrt{16 - 4(3 - 2h\lambda)}}{2(3 - 2h\lambda)}\right| < 1,$$

i.e.,

$$R = \left\{h\lambda \in C : \quad \left|\frac{2 \pm \sqrt{1 - 2h\lambda}}{3 - 2h\lambda}\right| < 1\right\}.$$

Problem 5. Suppose the difference scheme $u^{n+1} = Bu^n$ is stable, and $C(\Delta t)$ is a bounded family of operators. Show that the scheme

$$u^{n+1} = (B + \Delta t C(\Delta t))u^n$$

is stable.

Solution: Suppose $\|B^k\| \le K_1$ for $0 \le k \le n$ and $\|C(\Delta t)\| \le K_2$. It suffices to show $(B + \Delta t C(\Delta t))^n$ is bounded for $n\Delta t \le T$. This will consist of 2^n terms, of which $\left(\begin{array}{c} n \\ j \end{array}\right)$ terms involve j factors $\Delta t C$ interspersed in $n-j$ factors B; the latter can occur in at most $j+1$ sequences of consecutive factors, the norm of each sequence being bounded by K_1, and hence the norm of each such term by $K_2^j K_1^{j+1}$. Thus overall we obtain the bound

$$\begin{aligned}
\|(B + \Delta t C(\Delta t))^n\| &\le \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array}\right) K_1^{j+1} (\Delta t K_2)^j \\
&= K_1 (1 + \Delta t K_1 K_2)^n \\
&\le K_1 e^{n\Delta t K_1 K_2}
\end{aligned}$$

which is bounded for $n\Delta t \le T$.

Problem 6. Let A be an $m \times m$ nonsingular matrix. Suppose $\inf_{p_n \in P^n} ||p_n(A)|| = ||p^*(A)|| > 0$ where P^n denotes the set of all degree-n monic polynomials:

$$P^n = \{p : p \text{ is a polynomial of degree } n,\ p(z) = z^n + \cdots\}.$$

Prove that p^* is unique.

Solution: We prove by contradiction. Assuming there are two p_i for $i = 1, 2$ as minimizers, then $p = (p_1+p_2)/2$ shares the same 2-norm,

$$\|p_1\| = \|p_2\| = \|p\| = \sigma_1,$$

where σ_1 is the largest singular value. Let the SVD of p be

$$p(A) = U\text{diag}(\sigma_1, \ldots, \sigma_n)V^*.$$

Suppose σ_1 is $J-$fold, with left and right singular vectors $u_1, \ldots, u_J$ and $v_1, \ldots, v_J$, respectively.

By convexity of the 2-norm, we have

$$\sigma_1 = \|p(A)v_j\| \le \frac{1}{2}\left(\|p_1(A)v_j\| + \|p_2(A)v_j\|\right) \le \sigma_1,$$

which implies that

$$\|p_1(A)v_j\| = \|p_2(A)v_j\| = \sigma_1$$

and

$$(p_1 - p_2)v_j = 0, 1 \le j \le J.$$

Similarly we can show that $(p_1^* - p_2^*)u_j = 0$.

Construct $q \in P^n$ using $p_1 - p_2$ so that $qv_j = 0$ and $q^*u_j = 0$. For a fixed $\epsilon \in (0, 1)$, define

$$p_\epsilon = (1-\epsilon)p + \epsilon q \in P^n.$$

Hence

$$p_\epsilon^* p_\epsilon v_j = (1-\epsilon)p_\epsilon^* p(A)v_j = (1-\epsilon)\sigma_1 p_\epsilon^* u_j = (1-\epsilon)^2 \sigma_1^2 v_j.$$

This says that p_ϵ has right singular vector $v_1, \dots, v_J$ corresponding to the singular value $(1-\epsilon)\sigma_1$.

There are two cases to consider:

(1) $\|p_\epsilon\| = (1-\epsilon)\sigma_1 < \sigma_1$ is not the largest singular value, we see a contradiction.

(2) None of $v_1, \dots, v_J$ correspond to the largest singular value of p_ϵ. Using this fact and

$$p(A) = U\Sigma V^*,$$

we have

$$\begin{aligned}
\|p_\epsilon(A)\| &= \|p_\epsilon(A)V\| = \|p_\epsilon(A)[v_{J+1}, \dots, v_n]\| \\
&= \|(1-\epsilon)p(A)[v_{J+1}, \dots v_n] + \epsilon q(A)[v_{J+1}, \dots, v_n]\| \\
&\leq (1-\epsilon)\|[u_{J+1}, \dots u_n]\mathrm{diag}(\sigma_{J+1}, \dots, \sigma_n)\| + \epsilon\|q(A)[v_{J+1}, \dots, v_n]\| \\
&\leq (1-\epsilon)\sigma_{J+1} + \epsilon\|q(A)[v_{J+1}, \dots, v_n]\| \to \sigma_{J+1} < \sigma_J = \sigma_1
\end{aligned}$$

for ϵ small. This again leads to a contradiction. The uniqueness proof is complete.