S.-T. Yau College Student Mathematics Contests 2015

Algebra and Number Theory

Team

This exam contains 6 problems. Please choose 5 of them to work on.

Problem 1. (20pt) Let $V = \mathbb{R}^n$ be an Euclidean space equipped with usual inner product, and g an orthogonal matrix acting on V. For $a \in V$, let s_a denote the reflection

$$s_a(x) := x - 2\frac{(x,a)}{(a,a)}a, \qquad \forall x \in V.$$

(1.1) (10pt) For $a = (g-1)b \neq 0$, show that

$$\ker(s_a g - 1) = \ker(g-1) \oplus \mathbb{R}b.$$

(1.2) (10pt) Show that g is a product of $\dim[(g-1)V]$ reflections.

Problem 2. (20pt) Let p and q be two distinct prime numbers. Let G be a non-abelian finite group satisfying the following conditions:

1. all nontrivial elements have order either p or q;
2. The q-Sylow subgroup H_q is normal and is a nontrivial abelian group.

Show in steps the following statement:

> *The group G is of the form $(\mathbb{Z}/p\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^n$, where the action of $1 \in \mathbb{Z}/p\mathbb{Z}$ on $(\mathbb{Z}/q\mathbb{Z})^n \simeq \mathbb{F}_q^n$ is given by a matrix $M(1) \in \mathrm{GL}_n(\mathbb{F}_q)$ whose eigenvalues are all* primitive *p-th roots of unities.*

(2.1) (5pt) Let H_p denote a p-Sylow subgroup of G. Show that its inclusion into G induces an isomorphism $H_p \cong G/H_q$, and that $G \simeq H_p \ltimes H_q$.

(2.2) (5pt) Let $M : H_p \longrightarrow \mathrm{Aut}(H_q) \simeq \mathrm{GL}_n(\mathbb{F}_q)$ be the homomorphism induced from the conjugations. Show that for each $1 \neq a \in H_p$, $M(a)$ is semisimple whose eigenvalues are all *primitive* p-th roots of unities. In particular M is injective.

(2.3) (5pt) Show that if two nontrivial elements $a, b \in H_p$ commute with each other, then $a = b^n$ for some $n \in \mathbb{N}$, and that $H_p \simeq \mathbb{Z}/p\mathbb{Z}$.

(2.4) (5pt) Complete the solution of the problem.

Problem 3. (20pt) Let ζ be a root of unity satisfying an equation $\zeta = 1 + N\eta$ for an integer $N \geq 3$ and an algebraic integer η. Show that $\zeta = 1$.

Problem 4. (20pt) Let G be a finite group and (π, V) a finite dimensional $\mathbb{C}G$-module. For $n \geq 0$, let $\mathbb{C}[V]_n$ be the space of homogeneous polynomial functions on V of degree n. For a simple G-representation ρ, denote by $a_n(\rho)$ the multiplicity of ρ in $\mathbb{C}[V]_n$. Show that

$$\sum_{n\geq 0} a_n(\rho)t^n = \frac{1}{|G|}\sum_{g\in G} \frac{\overline{\chi_\rho(g)}}{\det(\mathrm{id}_V - \pi(g)t)}.$$

Problem 5. (20pt) Let A be an $n \times n$ complex matrix considered as an operator on $V = (\mathbb{C}^n, (\cdot,\cdot))$ with standard hermitian form. Let $A^* = \bar{A}^t$ be the hermitian transpose of A:

$$(Ax, y) = (x, A^*y), \qquad \forall x, y \in \mathbb{C}^n.$$

(5.1) (5pt) For any $\lambda \in \mathbb{C}$, show the identity:

$$\ker(A - \lambda)^\perp = (A^* - \bar{\lambda})V.$$

(5.2) (15pt) Show the equivalence of the following two statements:

- (a) A commutes with A^*;
- (b) there is a unitary matrix U (in the sense $U^* = U^{-1}$), such that UAU^{-1} is diagonal.

Problem 6. (20pt) Consider the polynomial $f(x) = x^5 - 80x + 5$.

(6.1) (5pt) Show that f is irreducible over $\mathbb{Q}$;

(6.2) (15 pt) Show in steps that the split field K of f has Galois group $G := \mathrm{Gal}(K/\mathbb{Q})$ isomorphic to S_5, the symmetric group of 5 letters.

- (a) (5pt) $f = 0$ has exactly two complex roots;
- (b) (5pt) G can be embedded into S_5 with image containing cycles (12345) and (12);
- (c) (5pt) $G \simeq S_5$.