GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Analysis and Differential Equations

Please solve 5 out of the following 6 problems.

1. Let $A = [a_{ij}]$ be a real symmetric $n \times n$ matrix. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, \cdots, x_n) = \exp(-\frac{1}{2}\sum_{i,j=1}^n a_{ij}x_ix_j)$. Prove that f is in $L^1(\mathbb{R}^n)$ if and only if the matrix A is positive definite.

Compute $\int_{\mathbb{R}^n} \exp(-\frac{1}{2}\sum_{i,j=1}^n a_{ij}x_ix_j + \Sigma_{i=1}^n b_ix_i)dx$ when A is positive definite.

2. Let V be a simply connected region in the complex plane and $V \neq \mathbb{C}$. Let a, b be two distinct points in V. Let ϕ_1, ϕ_2 be two one-to-one holomorphic maps of V onto itself. If $\phi_1(a) = \phi_2(a)$ and $\phi_1(b) = \phi_2(b)$, show that $\phi_1(z) = \phi_2(z)$ for all $z \in V$.

3. In the unit interval [0, 1] consider a subset
$E = \{x|$ in the decimal expansion of x there is no $4\}$,
show that E is measurable and calculate its measure.

4. Let $1 < p < \infty, L^p([0,1], dm)$ be the completion of $C[0,1]$ with the norm: $||f||p = (\int_0^1 |f(x)|^p dm)^{\frac{1}{p}}$, where dm is the Lebesgue measure. Show that $\lim_{\lambda\to\infty} \lambda^p m(x||f(x)| > \lambda) = 0$.

5. Let $\mathfrak{F} = \{e_\nu\}, \nu = 1, 2, ..., n$ or $\nu = 1, 2, ...$ is an orthonormal basis in an inner product space H. Let E be the closed linear subspace spanned by $\mathfrak{F}$. For any $x \in H$ show that the following are equivalent: 1) $x \in E$; 2) $||x||^2 = \Sigma_\nu |(x, e_\nu)|^2$; 3) $x = \Sigma_\nu (x, e_\nu)e_\nu$.

Let $H = L^2[0, 2\pi]$ with the inner product $< f, g >= \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx$,
$\mathfrak{F} = \{\frac{1}{\sqrt{2}}, \cos x, \sin x, ..., \cos nx, \sin nx, ...\}$
be an orthonormal basis. Show that the closed linear sub-space E spanned by $\mathfrak{F}$ is H.

6. Let $\mathcal{H} = L^2[0,1]$ relative to the Lebesgue measure and define $(Kf)(s) = \int_0^s f(t)dt$ for each f in $\mathcal{H}$. Show that K is a compact operator without eigenvalues.