S.-T. Yau College Student Mathematics Contests 2018

Analysis and Differential Equations

Team

Please solve the following 5 problems.

1. Suppose $\{f_n\}_{n=1}^{\infty} \in L^2(\mathbf{R})$ is a sequence that converges to 0 in the L^2 norm .
Prove that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 0$ almost everywhere.

2. Let $\hat{f}(\xi) = \int e^{-ix\xi} f(x)dx$ be the Fourier transform on Schwartz function $f \in S(\mathbf{R})$. Suppose $f \in S(\mathbf{R})$ satisfies $f(2\pi n) = 0$ and $\hat{f}(n) = 0$ for all integers n. Prove that $f = 0$.

3. If f is integrable on $\mathbf{R}^d$, then

$$\lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B f(y)dy = f(x),$$

for a.e. x, B is an open ball centered at x.

4. Let $C[0,1] = \{f : [0,1] \to \mathbf{R} |\ f \text{ is continuous}\}$ be the space of continuous function on $[0,1]$. Let $\rho(f,g) = \int_0^1 |f(x) - g(x)|dx$ be a metric on $[0,1]$.
Show that $(C[0,1], \rho)$ is not a complete metric vector space.
Construct a complete metric vector space $(W, \widetilde{\rho})$ such that $i : (C[0,1], \rho) \hookrightarrow (W, \widetilde{\rho})$ is an isometric embedding such that $\widetilde{\rho}|_{C[0,1]} = \rho$, $\overline{C[0,1]} = W$.

5. Let Ω be a simply connected domain in $\mathbf{C}$. Consider a point $z_0 \in \Omega$ and solve the Dirichlet problem in Ω with the boundary values $\log|\zeta - z_0|$. The solution is denoted by $G(z, z_0)$ and let $g(z, z_0) = G(z, z_0) - \log|z - z_0|$. Let $w = f(z) : \Omega \to D_1 = \{z||z| < 1\}$ be the one to one surjective conformal mapping with $f(z_0) = 0$. Show that
1) $g(z, z_0) = -\log|f(z)|$.
2) $g(z, z_0) = g(z_0, z)$. (Hint: Let $g(z, z_1) = g_1, g(z, z_2) = g_2$, calculate the integral $g_1 * dg_2 - g_2 * dg_1$ over the cycle $\partial\Omega - c_1 - c_2$, where c_1, c_2 are small circles around $z_1, z_2, du = u_x dx + u_y dy, *du = -u_y dx + u_x dy$.)

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Probability and Statistics

Team (5 problems)

Problem 1. Let X_i, $1 \le i \le N$ be i.i.d. random variables. Here X_1 is uniformly distributed on $[0,1]$. We reorder them as

$$\widetilde{X}_1 \le \widetilde{X}_2 \le \cdots \widetilde{X}_N$$

a) Let $N = 2m-1$, and $Y = \widetilde{X}_m$, please find the A and B such that

$$\frac{Y-A}{N^B}$$

has nontrivial distribution, and please find this distribution.

b) Let $N = 2m$, and $Y = \widetilde{X}_m - \widetilde{X}_{m-1}$, please find the A and B such that

$$\frac{Y-A}{N^B}$$

has nontrivial distribution, and please find this distribution.

Problem 2. Let $\mathbf{X} = (\mathbb{Z}_2)^{\mathbb{N}}$, i.e., $\mathbf{X} = (X_1, X_2 \cdots, X_N \cdots)$, $X_i \in (0,1)$. It can be considered as countable lightbulbs. 0 means off, 1 means on. We start with $\mathbf{X}_0 = \mathbf{0}$. Keep generating independent geometric random variables, whose distribution are $geom(1/2)$. Denote them as K_1, $K_2 \cdots$. Now let $\mathbf{X}_m$ (for $m \ge 1$) be as follows

$$(\mathbf{X}_m - \mathbf{X}_{m-1})_k = \mathbf{1}(k = K_m), \qquad \mathbb{Z}_2$$

i.e, in the $m-th$ turn, we only change the status of the K_m-th light bulb. Then what is the probability of all lights being off again, i.e.,

$$\mathbb{P}\left(\exists m > 1, \quad \mathbf{X}_m = \mathbf{0}\right)$$

Problem 3. Let $x_1, x_2, \ldots, x_n$ be d-dimensional vectors of real numbers with n sufficiently large but the exact value is not of importance.

A function of μ is defined to be

$$\ell(\mu) = \sup\{\sum_{i=1}^n \log p_i : \sum_{i=1}^n p_i x_i = \mu; \sum_{i=1}^n p_i = 1, p_1 > 0, \ldots, p_n > 0\}$$

on the space of the interior of the convex hull of $x_1, \ldots, x_n$.

(a) Show that this is a concave function of μ on the convex hull.

(b) Let $\bar{x} = n^{-1}\sum_{i=1}^n x_i$. Let $\mathbf{a}$ be a vector of length d. Prove that $\ell(\bar{x} + t\mathbf{a})$ is a decreasing function of t when $t > 0$.

Problem 4. Consider the histogram estimator, defined as follows. We observe *iid* random variables $X_1, \dots, X_n$, taking values in $[0,1]$ according to the distribution with PDF f (assuming it is sufficiently smooth). Define bins

$$B_1 = \left[0, \frac{1}{m}\right), B_2 = \left[\frac{1}{m}, \frac{2}{m}\right), \dots, B_m = \left[\frac{m-1}{m}, 1\right]$$

Let $h = 1/m$, v_j be the number of observations in bin B_j, and define $\hat{p}_j = v_j/n$ and $p_j = \int_{B_j} f(u)du$. Then the histogram estimator of the density f is

$$\hat{f}_n(x) = \sum_{j=1}^{m} \frac{\hat{p}_j}{h} I\{x \in B_j\}$$

1. Find the (exact) mean and variance of $\hat{f}_n(x)$.
2. Explain why increasing the number of bins decreases the bias of $\hat{f}_n(x)$.
3. If our goal is to minimize the mean-squared error

   $$MSE = E\left[\int (f(x) - \hat{f}_n(x))^2 dx\right],$$

   please give some advice on how to choose m.

Problem 5. Let $X_i \sim N(\theta_i, 1)$ independently for $i = 1, \dots, k$. We are interested in estimating $\tau = \theta_1^2 + \cdots + \theta_k^2$ given observations $X_1, \dots, X_k$.

1. A possible estimator of τ is $\tilde{\tau} = \sum_{i=1}^{k} X_i^2 - k$. Show that it is unbiased and compute its sampling variance.
2. Now assume the proper prior $\theta_i \sim N(0, A)$, independently for $i = 1, \dots, k$ and a given $A > 0$. Since A is unknown, please provide an estimator $\hat{A}$ of A and also derive the empirical Bayes estimator of τ, denoted as $\hat{\tau}_B$. (Hint: $\hat{\tau}_B = E(\tau \mid X_1, \dots, X_k, \hat{A})$).
3. How do you compare the two estimators, $\tilde{\tau}$ and $\hat{\tau}_B$?

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Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Let X be $(S^2 \times S^2) \cup_{S^2} D^3$, where we attach the 3-disk via the map

$$S^2 \to S^2 \vee S^2$$

which crushes a great circle connecting the north and south poles. Compute the homology groups of X.

2. (a) Let A be a single circle in $\mathbb{R}^3$. Compute the fundamental group $\pi_1(\mathbb{R}^3 - A)$.

(b) Let A and B be disjoint circles in $\mathbb{R}^3$, supported in the upper and lower half space, respectively. Compute $\pi_1(\mathbb{R}^3 - (A \cup B))$.

3. Consider the differential 1-form $\omega = xdy - ydx + dz$ in $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

4. Show that

$$Q^n := \{(x^1, \cdots, x^{n+1}) \in \mathbb{R}^{n+1}; \sum_{i=1}^{n+1} (x^i)^4 = 1\}$$

is a differentiable manifold.

5. Let M be a closed surface in $\mathbb{R}^3$. Prove that

$$\int_M |K| d\sigma \geq 4\pi(1 + g),$$

where K, g and $d\sigma$ is the Gaussian curvature, the genus and the area element of M, respectively.

6. Let M be an n-dimensional compact and simply connnected Riemannian manifold. If the sectional curvature K_M of M satisfies

$$\frac{1}{4} < K_M \leq 1,$$

then M is homeomorphic to S^n.

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Algebra and Number Theory

Team

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.** Recall that a ring E is said to be *local* if for every $u \in E$, at least one of the elements u and $1-u$ is invertible. Let R be a ring and let M be an R-module.

(a) (8 points) Show that if $\mathrm{End}_R(M)$ is a local ring, then M is indecomposable.

(b) (12 points) Assume M indecomposable and of finite length. Prove the Fitting lemma: Every endomorphism u of M is either invertible or nilpotent. Deduce that $\mathrm{End}_R(M)$ is a local ring.

Problem 2 (20 points)**.**

(a) (6 points) Let $n \geq 2$ be an integer. Show that there exists an integer m with $1 \leq m \leq n-1$ such that the binomial coefficient $\binom{n}{m}$ satisfies $\binom{n}{m} \geq 2^n/n$.

(b) (6 points) Let $0 \leq m \leq n$ be integers with $n \geq 1$. Show that for every prime number p,

$$v_p\left(\binom{n}{m}\right) \leq \log_p(n)$$

Here v_p is the p-adic valuation: $v_p(p^a b) = a$ for integers b prime to p and $a \geq 0$.

(c) (8 points) Let $n \geq 2$ be an integer and let $\pi(n)$ denote the number of prime numbers $p \leq n$. Deduce the following inequality of Chebyshev:

$$\pi(n) \geq \frac{n}{\log_2 n} - 1.$$

Problem 3 (20 points)**.** Let $n \geq 1$ be an integer and let $\Phi_n(X) \in \mathbb{Q}[X]$ denote the n-th cyclotomic polynomial, i.e.

$$\Phi_n(X) := \prod_{\xi} (X - \xi),$$

where ξ runs through primitive n-th roots of unity in $\mathbb{C}$. Recall that $X^n - 1 = \prod_{d|n} \Phi_d(X)$ and $\Phi_n(X)$ belongs to $\mathbb{Z}[X]$. Let p be a prime number such that $p \nmid n$. Denote by $\overline{\Phi}_n$ the residue class of Φ_n in $\mathbb{F}_p[X]$. Prove the following statements:

(a) (8 points) The roots of $\overline{\Phi}_n = 0$ in the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ are exactly the *primitive* n-th roots of 1 in $\overline{\mathbb{F}}_p$.

(b) (12 points) $\overline{\Phi}_n$ is irreducible in $\mathbb{F}_p[X]$ if and only if $(\mathbb{Z}/n\mathbb{Z})^\times$ is a cyclic group generated by the class of p.

Problem 4 (20 points)**.** Let G be a finite group. Let V be a finite-dimensional complex representation of G and let $\chi\colon V \to \mathbb{C}$ be the associated character.

(a) (8 points) Show that there exists a subfield $L \subseteq \mathbb{C}$ containing the image of χ such that $L/\mathbb{Q}$ is a finite Galois extension. Show moreover that

$$B(\chi) = \prod_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} \prod_{g \in G} \sigma(\chi(g))$$

belongs to $\mathbb{Z}$.

(b) (12 points) Suppose that χ is irreducible and $\dim(V) \geq 2$. Show that there exists $g \in G$ with $\chi(g) = 0$. (*Hint.* One may apply the inequality of arithmetic and geometric means to $|B(\chi)|^2$.)

Problem 5 (20 points)**.** Let F be a field, V an F-vector space of dimension d and $W \subseteq V$ a subspace. Let $f\colon W \to V$ be an F-linear map. Assume that the only subspace $W' \subseteq W$ such that $f(W') \subseteq W'$ is $\{0\}$.

(a) (6 points) Let $v \in V$ be a non-zero vector. Show that there exists a unique integer $k(v) \geq 0$ such that $v, f(v), f^2(v), \ldots, f^{k(v)-1}(v) \in W$ but $f^{k(v)}(v) \notin W$. Show moreover that $v, f(v), \ldots, f^{k(v)}(v)$ are linearly independent over F.

(b) (14 points) Prove that given $\lambda_1, \ldots, \lambda_d \in F$, there exists an F-linear extension of f to $\tilde{f}\colon V \to V$ such that the characteristic polynomial of $\tilde{f}$ is $\prod_{i=1}^{d}(\lambda - \lambda_i)$. (*Hint.* You may first treat the special case $V = \bigoplus_{i=0}^{k(v)} F f^i(v)$. For the general case, consider the subset $W_n \subseteq V$ of vectors $v \in V$ with $k(v) \geq n$ or $v = 0$.)

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Applied Math. and Computational Math.

Team (5 problems)

1. Let H be a bipartite graph with the bipartition $V = V_1 \cup V_2$, where $|V_1| = |V_2| = n$. We say that H satisfies the (p, q)-condition if (i) for all subsets $I \subseteq V_1$ of cardinality at most p, the inequality $|I| \leq |N(I)|$ holds, and (ii) for all subsets $J \subseteq V_2$ of cardinality at most q, the inequality $|J| \leq |N(J)|$ holds. Note that the $(n, 0)$-condition is Hall's original condition in his marriage theorem.

Prove that if H satisfies the (p, q)-condition with $n \leq p + q$, then H contains a matching of size n.

2. Let C_n be the n dimensional hypercube, i.e., the graph whose vertex set V is $\{0, 1\}^n$, and whose edges are defined by: two vertices $u = u_1u_2 \ldots u_n$ and $v = v_1v_2 \ldots v_n$ are adjacent iff $u_i \neq v_i$ for exactly one $i \in [n]$. Let $\mathbb{R}[V]$ be the vector space of all the functions $f : V \to \mathbb{R}$. The space $\mathbb{R}[V]$ has a natural inner product. For $f, g \in \mathbb{R}[V]$,

$$< f, g >= \sum_{u \in \{0,1\}^n} f(u)g(u).$$

The standard basis of $\mathbb{R}[V]$ is the set $\{f_u : u \in \{0, 1\}^n\}$ where $f_u(v) = \delta_{u,v}$, the Kronecker delta, for $u, v \in \{0, 1\}^n$. Denote by B_1 the standard basis.

(1) For any two vertices $u, v \in \{0, 1\}^n$, $u \cdot v$ is defined to be $\sum_i u_iv_i$. For each $u \in \{0, 1\}^n$, define a function $\chi_u \in \mathbb{R}[V]$ by letting

$$\chi_u(v) = (-1)^{u \cdot v}.$$

Prove that the set $\{\chi_u : u \in \{0, 1\}^n\}$ is orthogonal with respect to the inner product of $\mathbb{R}[V]$, i.e.,

$$< \chi_u, \chi_v >= \delta_{u,v}2^n,$$

for all $u, v \in \{0, 1\}^n$.

(2) Prove that the set $\{\chi_u : u \in \{0, 1\}^n\}$ forms a basis of the vector space $\mathbb{R}[V]$. Denoted by B_2 this basis.

(3) For $1 \leq i \leq n$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \{0, 1\}^n$ where the only 1 occurs in position i. Let $S = \{e_1, e_2, \ldots, e_n\}$.

Define a linear transformation $\Phi : \mathbb{R}[V] \to \mathbb{R}[V]$ as follows. For $f \in \mathbb{R}[V]$, Φf is the element in $\mathbb{R}[V]$ which is given by

$$(\Phi f)(v) = \sum_{e_i \in S} f(v + e_i)$$

where $v + e_i$ is the usual vector addition modulo 2.

Prove that the matrix of Φ with respect to the standard basis B_1 is just $A(C_n)$, the adjacency matrix of the hypercube C_n.

(4) Prove that $\Phi\chi_u = \lambda_u\chi_u$ for each $u \in \{0,1\}^n$, where

$$\lambda_u = \sum_{e\in S}(-1)^{u\cdot e} = n - 2|u|,$$

where $|u|$ is the number of 1's in $u = u_1u_2\dots u_n$.

(5) Compute the eigenvalues of the matrix $A(C_n)$.

3. Let $A \in \mathbb{R}^{n\times n}$, and assume that there are unitary matrix Q and diagonal matrix $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ such that $A = QDQ^*$. Let E_k be the space spanned by the first k columns of Q. We let

$$\widehat{P} = \begin{pmatrix} I_k & \\ & 0 \end{pmatrix}, \quad P = Q\widehat{P}Q^*$$

where I_k is the $k\times k$ identity matrix.

(1) Show that P is an orthogonal projection onto E_k.

(2) Assume that

$$|\lambda_1| \ge \cdots \ge |\lambda_k| > |\lambda_{k+1}| \ge \cdots \ge |\lambda_n|.$$

Let $X^{(0)} \in \mathbb{R}^{n\times k}$ and assume $PX^{(0)}$ is injective. We define the iterations

$$X^{(m+1)} = AX^{(m)}.$$

Show that there is a matrix $\Lambda \in \mathbb{R}^{k\times k}$ such that

$$\frac{\|(AX^{(m)} - X^{(m)}\Lambda)y\|}{\|PX^{(m)}y\|} \le \Big(\frac{|\lambda_{k+1}|}{|\lambda_k|}\Big)^m \frac{\|(AX^{(0)} - X^{(0)}\Lambda)y\|}{\|PX^{(0)}y\|}, \quad \forall y \in \mathbb{R}^k\backslash\{0\}.$$

4. For the one-way equation

$$u_t + au_x = f, \tag{1}$$

consider the multistep scheme given by

$$\frac{3u_m^{n+1} - 4u_m^n + u_m^{n-1}}{2k} + a\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} = f_m^{n+1}. \tag{2}$$

(1) Show that the scheme is second order accurate.

(2) Show that the scheme is unconditionally stable.
(Hint: (i) apply von Neumann analysis to the scheme with $f \equiv 0$ and find the characteristic polynomial. (ii) show that for all k, h, the characteristic polynomial satisfies the root condition: all roots reside in the unit disk, and all roots on the unit circle are simple. (iii) for a root r of the characteristic polynomial, it would be more convenient to study the form $\frac{1}{r} = X + iY$ and prove that $X^2 + Y^2 \ge 1$.)

5. For a convex function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$ is convex and open, define a subgradient of f at $x_0 \in D$ to be any vector $s \in \mathbb{R}^n$ such that

$$f(x) - f(x_0) \geq s \cdot (x - x_0)$$

for all $x \in D$. The subgradient is a plausible choice for generalizing the notion of a gradient at a point where f is not differentiable. The subdifferential $\partial f(x_0)$ is the set of all subgradients of f at x_0.

(1) What is $\partial f(0)$ for the function $f(x) = |x|$.

(2) Suppose we wish to minimize a convex and continuous function $f : \mathbb{R}^n \to \mathbb{R}$, which may not differentiable everywhere. Propose an optimality condition involving subdifferential for a point x_* to be a minimizer of f. Show that your condition holds if and only if x_* is a globally minimizer f.

(3) The *subgradient method* extends the gradient descent to a wider class of functions. Analogously to the gradient descent, the subgradient method performs the iteration

$$x_{k+1} = x_k - \alpha g_k,$$

where $\alpha > 0$ is small stepsize that is known as the learning rate, and g_k is *any* subgradient of f at x_k. This method might not decrease f in each iteration, so instead we keep track of the best iterate we have seen so far, x_k^{best}.

In the following parts, assume that f is Lipschitz continuous with constant $L > 0$, $\|x_1 - x_*\|_2 \leq B$ for some $B > 0$. Under these assumptions we will show that

$$\lim_{k\to\infty} f(x_k^{\text{best}}) \leq f(x_*) + \frac{L^2}{2}\alpha, \tag{3}$$

a bound characterizing convergence of the subgradient method.

(a) Derive an upper bound for the error $\|x_{k+1} - x_*\|_2^2$ of x_{k+1} in terms of $\|x_k - x_*\|_2^2$, g_k, α, $f(x_k)$ and $f(x_*)$.

(b) By recursively applying the result from Problem 3a, provide an upper bound for $\|x_{k+1} - x_*\|_2^2$.

(c) Incorporate $f(x_k^{\text{best}})$ into your upper bound in Problem 3b, and take a limit as $k \to \infty$ to obtain the desired convergence result (3).

(d) Suggest a best choice of the learning rate α.