S.-T. Yau College Student Mathematics Contests 2021

Analysis and Differential Equations

Solve every problem.

Problem 1. Prove that $f(x) \equiv 0$ is the only solution in $L^2(\mathbf{R^n})$ such that

$$\triangle f = 0.$$

Solution: Consider the Fourier transform of the equation $\triangle f = 0$. This yields

$$-|\xi|^2 \widehat{f}(\xi) = 0.$$

By the Plancherel Theorem, $f \in L^2(\mathbf{R^n})$ implies that $\widehat{f} \in L^2(\mathbf{R^n})$. Therefore, the above equation shows supp $\left(\widehat{f}\right) \subset \{0\}$. Hence $\widehat{f} = 0$ in L^2. Hence, $f = 0$ in L^2. Since harmonic functions are smooth, $f \equiv 0$.

Problem 2. Let $X \subset C([0,1])$ be a finite dimensional linear subspace of the space of real-valued continuous functions on $[0,1]$. Show that, for a sequence of functions $\{f_k\}_{k\geqslant 1} \subset X$, if it converges pointwise, it converges uniformly.

Solution: Let $\varphi_1, \ldots, \varphi_n$ be a basis of X. We first show that there exists $t_1, t_2, \ldots, t_n \in [0,1]$ so that $\det\left(\varphi_i(t_j)\right) \neq 0$, where $1 \leqslant i, j \leqslant n$. Consider the linear functionals

$$\ell_t : X \longrightarrow \mathbb{R}, \quad f \longmapsto f(t).$$

We have $\cap_{t\in[0,1]}\ker(\ell_t) = \{0\}$. Therefore, there exists $t_1, t_2, \ldots, t_n \in [0,1]$ so that $\cap_{i\leqslant n}\ker(\ell_{t_i}) = \{0\}$. This means that $\det\left(\varphi_i(t_j)\right) \neq 0$.

We write $\{f_k\}_{k\geqslant 1}$ in terms of our basis:

$$f_k(x) = \sum_{j=1}^{n} \alpha_j^{(k)} \varphi_j(x).$$

Therefore,

$$\begin{pmatrix} f_k(x_1) \\ f_k(x_2) \\ \vdots \\ f_k(x_n) \end{pmatrix} = \left(\varphi_i(x_j)\right) \begin{pmatrix} \alpha_1^{(k)} \\ \alpha_2^{(k)} \\ \vdots \\ \alpha_n^{(k)} \end{pmatrix} = A \begin{pmatrix} \alpha_1^{(k)} \\ \alpha_2^{(k)} \\ \vdots \\ \alpha_n^{(k)} \end{pmatrix}.$$

We obtain that

$$\begin{pmatrix} \alpha_1^{(k)} \\ \alpha_2^{(k)} \\ \vdots \\ \alpha_n^{(k)} \end{pmatrix} = A^{-1} \begin{pmatrix} f_k(x_1) \\ f_k(x_2) \\ \vdots \\ f_k(x_n) \end{pmatrix}.$$

Since $\{f_k\}_{k\geqslant 1}$ converges pointwise, it converges on $x_1, \ldots, x_n$. Therefore,$\{\alpha_i^{(k)}\}_{k\geqslant 1}$ converges to some α_i. This implies that $\{f_k\}_{k\geqslant 1}$ converges uniformly to $\alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n$.

Problem 3.

(a) For $f \in L^1(\mathbf{R^n})$, $g \in L^\infty(\mathbf{R^n})$, show that their convolution $f * g$ is a well-defined continuous function.

(b) Let $E \subset \mathbf{R^n}$ be a Lebesgue measurable set with Lebesgue measure $m(E) > 0$. Prove that

$$E - E := \left\{x - y \,\middle|\, x \in E, y \in E\right\}$$

contains an open neighborhood of $0 \in \mathbf{R^n}$.

Solution:

(a) This is standard: In fact, we have $\|f * g\|_{L^\infty} \leqslant \|f\|_{L^1}\|g\|_{L^\infty}$. Therefore, by the continuity argument, it suffices to prove the theorem for $f \in C_0^\infty(\mathbf{R^n})$. In this case, we have

$$\begin{aligned}|f * g(x_0 + x) - f * g(x_0)| &= \left|\int_{\mathbf{R^n}} \big(f(x_0 + x - y) - f(x_0 - y)\big)g(y)\,dy\right| \\ &\leqslant \|g\|_{L^\infty} \int_{\mathbf{R^n}} \big|f(x_0 + x - y) - f(x_0 - y)\big|\,dy\end{aligned}$$

Now let $x \longrightarrow 0$, the integrand converges to 0 uniformly. This yields **(a)**.

(b) It suffices to consider the case where $m(E) < \infty$. We take $f = \mathbf{1}_E$, $g = \mathbf{1}_{-E}$, thus $h(x) = f * g$ is a continuous function. In particular, $h(0) = m(E) > 0$. Therefore, there exists an open set U such that $0 \in U$ and $h\big|_u > \delta > 0$ for some $\delta > 0$. For $x \in U$, by definition,

$$h(z) = \int_{\mathbf{R^n}} \mathbf{1}_E(x - y)\mathbf{1}_{-E}(y)dy > 0.$$

Therefore, there must be some $y \in -E$, such that $x - y = x + (-y) \in E$. This implies $x \in E - (-y) \subset E - E$. Hence $U \subset E - E$.

Problem 4. Assume that P is a polynomial with complex coefficients. Prove that there exists infinitely many solutions of the following equations on $\mathbf{C}$:

$$e^z = P(z).$$

Solution: This is an application of big Picard's theorem at 0.

Problem 5. Let f be a bounded holomorphic function defined on $B = \left\{z \,\middle|\, 0 < \mathrm{Re}(z) < 1\right\}$ that can be extended as a continuous function on $\overline{B}$. Let

$$A_0 = \sup_{\mathrm{Re}(z)=0} |f(z)| > 0, \;\; A_0 = \sup_{\mathrm{Re}(z)=1} |f(z)| > 0.$$

Prove that for all $z \in B$, we have

$$|f(z)| \leqslant (A_0)^{1-\mathrm{Re}(z)}(A_1)^{\mathrm{Re}(z)}.$$

Solution: We consider the function $g(z) = f(z)(A_0)^{z-1}(A_1)^{-z}$. This is a holomorphic function defined on B and bounded by 1. We consider the function $h(z) = g(z)e^{\varepsilon z^2}$. This function is bounded for $z \longrightarrow \pm i\infty$, therefore, it is bounded by its maximal value on the boundary. Letting $\varepsilon \longrightarrow 0$ proves the statement.

Problem 6. Assume that $\Omega \subset \mathbf{R^n}$ is a bounded domain with smooth boundary. Prove that there exists a positive constant ε_0 so that for all real numbers $\varepsilon < \varepsilon_0$, for all $f \in L^2(\Omega)$, there exist a unique $u \in H_0^1(\Omega)$ so that

$$-\triangle u + \varepsilon \sin(u) = f$$

in the sense of distributions.

Solution: We consider the functional

$$E(u) = \int_\Omega \tfrac{1}{2} |\nabla u|^2 - \cos(u) - fu$$

defined on $H_0^1(\Omega)$. $E(u)$ is bounded below by Poincaré's inequality. Therefore, a minimizing sequence gives a solution. To show that the solution is unique, we assume that $u_1, u_2 \in H_0^1(\Omega)$ so that

$$-\triangle u_i + \varepsilon \sin(u_i) = f \implies -\triangle(u_1 - u_2) + \varepsilon\,(\sin(u_1) - \sin(u_2)) = 0.$$

We multiply the equation by $u_1 - u_2$ and the integrate by parts, this leads to

$$\left\|\nabla(u_1 - u_2)\right\|_{L^2}^2 = \varepsilon \left| \int_\Omega (u_1 - u_2)(\sin(u_1) - \sin(u_2)) \right| \leqslant \varepsilon \left| \int_\Omega |u_1 - u_2||u_1 - u_2| \right|.$$

Therefore,

$$\left\|\nabla(u_1 - u_2)\right\|_{L^2}^2 \leqslant \varepsilon \left\|u_1 - u_2\right\|_{L^2}^2.$$

If $\varepsilon_0 < \lambda_1(\Omega)$, the Poincaré inequality implies that

$$\left\|\nabla(u_1 - u_2)\right\|_{L^2}^2 \leqslant \frac{\varepsilon}{\lambda_1(\Omega)} \left\|\nabla(u_1 - u_2)\right\|_{L^2}^2.$$

Hence, $u_1 = u_2$.