S.-T. Yau College Student Mathematics Contests 2019

Applied and Computational Math

Individual (4 problems)

1) (20 points)

Given a set $\mathcal{X}$, $m \in \mathbb{N}$ and a hypothesis space $\mathcal{H}$, define

$$\Pi_{\mathcal{H}}(m) = \max_{\{x_1, x_2, \ldots, x_m\} \subseteq \mathcal{X}} |\{(h(x_1), h(x_2), \ldots, h(x_m)) | h \in \mathcal{H}\}|$$

where $|\mathcal{S}|$ denotes the cardinality of the set $\mathcal{S}$. The VC dimension of $\mathcal{H}$ is

$$\mathrm{VC}(\mathcal{H}) = \max\{m : \Pi_{\mathcal{H}}(m) = 2^m\}.$$

(i) Let $\mathcal{X} = \mathbf{R}$. If $a \leqslant b$, define $h(x; a, b) = 1$ if $x \in [a, b]$ and $h(x) = -1$ if $x \notin [a, b]$. Find the VC dimension of the hypothesis space $\mathcal{H} = \{h(x; a, b) | a, b \in \mathbf{R}, a \leqslant b\}$.
(ii) Let $\mathcal{X} = \mathbf{R}^d$, $\mathcal{H}$ to be the set of linear classifiers, i.e. $\mathcal{H} = \{f(x) | f(x) = \mathrm{sign}(w^\top x + b), w \in \mathbf{R}^d, b \in \mathbf{R}\}$ where $\mathrm{sign}(x) = 1$ if $x > 0$, $\mathrm{sign}(x) = -1$ if $x < 0$ and $\mathrm{sign}(x) = 0$ if $x = 0$. Show that the VC dimension of $\mathcal{H}$ is $d + 1$.

2) (25 points)

Consider Richardson's difference scheme for the heat equation $u_t = u_{xx}$:

$$\frac{1}{2k}\left(u(x, t+k) - u(x, t-k)\right) = \frac{1}{h^2}\left(u(x-h, t) - 2u(x, t) + u(x+h, t)\right).$$

(i) Show that this scheme has second-order truncation error.
(ii) Use either ODE principles or von Neumann analysis to show that this scheme is unconditionally unstable.
(iii) Demonstrate a minor modification of the left-side of Richardson's scheme that yields a familiar unconditionally stable scheme and prove it.

3) (25 points)

Let $\emptyset \neq K$ be a closed convex set in $\mathbf{R}^n$, i.e., K is a closed set and for any $x, y \in K$ and $\lambda \in (0, 1)$, $\lambda x + (1 - \lambda) y \in K$. For any $z \in \mathbf{R}^n$, let $\Pi_K(z)$ denote the metric projection of z onto K, which is the unique optimal solution of following problem:

$$\min \frac{1}{2}\|y - z\|_2^2, \quad \text{s.t.} \quad y \in K. \tag{1}$$

Show that
(i) the point $y \in K$ solves (1) if and only if

$$(z - y)^T (d - y) \leqslant 0, \quad \forall\, d \in K;$$

(ii) for any $y, z \in \mathbf{R}^n$,

$$\|\Pi_K(y) - \Pi_K(z)\|_2 \leq \|y - z\|_2;$$

(iii) $\Theta(\cdot)$ is continuously differentiable with its gradient given by

$$\nabla \Theta(z) = z - \Pi_K(z),$$

where for any $z \in \mathbf{R}^n$, $\Theta(z) := \frac{1}{2}\|z - \Pi_K(z)\|_2^2$.

4) (25 points) The scientists FitzHugh (1961) and Nagumo, Arimoto, Yoshizawa (1962) derived a mathematical model to characterize the behavior of a neuron under the externally injected current I:

$$\begin{cases} \frac{dV}{dt} & = V - \frac{1}{3}V^3 - W + I, \\ \frac{dW}{dt} & = \frac{1}{\tau}\big(V + a - bW\big), \end{cases}$$

where the variable V describes the membrane potential of the neuron, the variable W describes the current arising from opening and closing of ion channels on the neurons membrane. The variables τ, a and b are parameters with typical values: $a = 0.7$,$b = 0.8$ and $\tau = 13$.

(i) For a small positive constant current I, how the neuron behaves.

(ii) For a large positive constant current I, how the neuron behaves.

(iii) Suppose one injects a pulse current with different magnitude at some time t_0, i.e., $I = I_0\delta(t - t_0)$, where I_0 describes the magnitude of the pulse, analyze the dynamical behavior of the neuron when I_0 is small or large.