S.-T. Yau College Student Mathematics Contests 2024

Computational and Applied Mathematics

1. Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix. Let $u,\ v \in \mathbb{R}^n$ be column vectors. Define the rank 1 perturbation $\widehat{A} = A + uv^T$.

   (a) Derive a necessary and sufficient condition for $\widehat{A}$ to be invertible.

   (b) Let x, z and b be column vectors in $\mathbb{R}^n$. Suppose one can solve $Az = b$ with $\mathcal{O}(n)$ floating-point operations (flops). Under the conditions derived in(a), design an algorithm to solve $\widehat{A}x = b$ with $\mathcal{O}(n)$ flops, and provide justification for your answer.

2. Consider the integral

$$\int_0^\infty f(x)\,dx$$

   where f is continuous, $f'(0) \neq 0$, and $f(x)$ decays like $x^{-1-\alpha}$ with $\alpha > 0$ in the limit $x \to \infty$.

   (a) Suppose you apply the equispaced composite trapezoid rule with n subintervals to approximate

$$\int_0^L f(x)\,dx.$$

   What is the asymptotic error formula for the error in the limit $n \to \infty$ with L fixed?

   (b) Suppose you consider the quadrature from (a) to be an approximation to the full integral from 0 to ∞. How should L increase with n to optimize the asymptotic rate of total error decay? What is the rate of error decrease with this choice of L? 5

   (c) Make the following change of variable $x = \dfrac{L(1+y)}{1-y}$, $y = \dfrac{x-L}{x+L}$ in the original integral to obtain

$$\int_{-1}^1 F_L(y)\,dy.$$

   Suppose you apply the equispaced composite trapezoid rule; what is the asymptotic error formula for fixed L?

   (d) Depending on α, which method - domain truncation or change-of-variable - is preferable?

3. Consider the Chebyshev polynomial of the first kind

$$T_n(x) = \cos(n\theta), \quad x = \cos(\theta), \quad x \in [-1, 1].$$

The Chebyshev polynomials of the second kind are defined as

$$U_n(x) = \frac{1}{n+1} T'(x), \quad n \geq 0.$$

(a) Derive a recursive formula for computing $U_n(x)$ for all $n \geq 0$.

(b) Show that the Chebyshev polynomials of the second kind are orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \sqrt{1 - x^2}\, dx$$

(c) Derive the 2-point Gaussian Quadrature rule for the integral

$$\int_{-1}^{1} f(x) \sqrt{1 - x^2}\, dx = \sum_{j=1}^{2} w_j f(x_j)$$

4. Consider the boundary value problem

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = f(x), \quad u(0) = u(1) = 0$$

where $a(x) > \delta \geq 0$ is a bounded differentiable function in $[0, 1]$. We assume that, although $a(x)$ is available, an expression for its derivative, $\frac{da}{dx}$, is not available.

(a) Using finite differences and an equally spaced grid in $[0, 1]$, $x_l = hl, l = 0, \ldots, n$ and $h = 1/n$, we discretize the ODE to obtain a linear system of equations, yielding an $O(h^2)$ approximation of the ODE. After the application of the boundary conditions, the resulting coefficient matrix of the linear system is an $(n-1)\times(n-1)$ tridiagonal matrix.

Provide a derivation and write down the resulting linear system (by giving the expressions of the elements).

(b) Utilizing all the information provided, find a disc in $\mathbb{C}$, the smaller the better, that is guaranteed to contain all the eigenvalues of the linear system constructed in part (a).

5. (a) Verify that the PDE
$$u_t = u_{xxx}$$
is well posed as an initial value problem.

(b) Consider solving it numerically using the scheme
$$\frac{u(t+k,x) - u(t-k,x)}{2k} = \frac{-\frac{1}{2}u(x-2h,t) + u(x-h,t) - u(x+h,t) + \frac{1}{2}u(x+2h,t)}{h}.$$
Determine this scheme's stability condition.

6. Consider the diffusion equation
$$\frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2}, \quad v(x,0) = \phi(x), \quad \int_a^b v(x,t)\,dx = 0$$
with $x \in [a,b]$ and periodic boundary conditions. The solution is to be approximated using the central difference operator L for the 1D Laplacian.
$$Lv_m = \frac{v_{m+1} - 2v_m + v_{m-1}}{h^2},$$
and the following two finite different approximations, (i) Forward-Euler
$$v_{n+1} = v_n + \mu k L v_n, \tag{1}$$
and (ii) Crank-Nicolson
$$v_{n+1} = v_n + \mu k (L v_n + L v_{n+1}). \tag{2}$$
Throughout, consider $[a,b] = [0, 2\pi]$ and the finite difference stencil to have periodic boundary conditions on the spatial lattice $[0, h, 2h, \ldots, (N-1)h]$ where $h = \dfrac{2\pi}{N}$ and N is even.

(a) Determine the order of accuracy of the central difference operator Lv in approximating the second derivative v_{xx}.

(b) Using $v_m^n = \displaystyle\sum_{l=0}^{N-1} \hat{v}_l^n \exp\left(-i\frac{2\pi l m}{N}\right)$ give the updates $\hat{v}_l^{n+1}$ in terms of $\hat{v}_l^n$ for each of the methods, including the case $l = 0$.

(c) Give the solution for v_m^n for each method when the initial condition is $\phi(m\Delta x) = (-1)^m$.

(d) What are the stability constraints on the time step k for each of the methods, if any, in equations (1) and (2)? Show there are either no constraints or express them in the form $k \leq F(h, \mu)$.