S.-T. Yau College Student Mathematics Contests 2019

Analysis and Differential Equations

Individual (5 problems)

1) Let $F\colon \mathbb{R} \to \mathbb{R}$ be a strictly convex function. Let $u\colon [0,1] \to \mathbb{R}$ be a continuous function, with

$$\int_0^1 u(x)\,dx = 0.$$

Show that

$$\int_0^1 F(u(x))\,dx \leqslant \frac{F(\|u\|_\infty) + F(-\|u\|_\infty)}{2}$$

where $\|u\|_\infty := \sup_{x\in[0,1]} |u(x)|$. Also determine when equality occurs.

2) Prove that there exists a universal constant K, for all C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$, if $f \in L^1\left(\mathbb{R}^2\right) \cap L^2\left(\mathbb{R}^2\right)$ and $|\nabla f| \in L^2(\mathbb{R}^2)$, we have the following inequality:

$$\|f\|^2_{L^2(\mathbb{R}^2)} \leqslant K \|f\|_{L^1(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)}.$$

Can you provide constant K so that $K < 10$? In the problem, all the L^p-spaces are defined with respect to the Lebesgue measure.

3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function. Suppose

$$\lim_{|x|\to\infty} \frac{|f(x)|}{\ln|x|} = 0.$$

Prove or disprove that f is a constant.

4) (a) Show that there does not exist a holomorphic function f on $\mathbb{C} \setminus \{1,-1\}$ so that

$$f'(z) = \frac{1}{(z^2-1)^{2019}} \quad \text{for all } z \in \mathbb{C} \setminus \{1,-1\}.$$

(b) Show that there exist a set $L \subset \mathbb{C}$ and a holomorphic function F on $\mathbb{C} \setminus L$ so that L has Hausdorff dimension 1, and

$$F'(z) = \frac{1}{(z^2-1)^{2019}} \quad \text{for all } z \in \mathbb{C} \setminus L.$$

5) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Prove that, for all $p > 1$ and $1 \leqslant q < \infty$, for all $f \in L^p(\Omega)$, there exists a unique $u \in H^1_0(\Omega)$, such that

$$\triangle u = |u|^{q-1}u + f \text{ in } \Omega.$$