S.-T. Yau College Student Mathematics Contests 2013

Applied Math. and Computational Math.

Individual

Please solve as many problems as you can!

1. We consider the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^3 \times \mathbb{R}_+$.

(a): (5 pts) A right going pulse with speed 1

$u(x,y,z,t) = 1$ for $t < x < t+1$; $\quad u(x,y,z,t) = 0$ else

is clearly a solution to the wave equation. However, it is a discontinuous solution, explain in which sense it is a solution to the equation.

(b): (5 pts) Surprisingly, one can construct smooth progressive wave solutions with speed larger than 1. In astronomy this kind of wave known as superluminal wave. Try a solution of the form

$$u(x,y,z,t) = v(\frac{x-ct}{\sqrt{c^2-1}}, y, z), \quad c \in \mathbb{R}^3, \quad |c| > 1.$$

Derive an equation for v and show that there is a nontrivial solution with compact support in (y,z) for any fixed x, t.

(c): (5 pts) For any $R > 0$, $0 < t < R$, show that energy

$$E(t) := \int_{|\vec{x}| \le R-t} \left(|u_t(\cdot,t)|^2 + |\nabla u(\cdot,t)|^2 \right) d\vec{x}$$

is a decreasing function.

(c): (10 pts) Show that smooth superluminal progressive wave solutions of the form

$$u(\vec{x},t) = v(\vec{x} - \vec{c}t), \vec{c} \in \mathbb{R}^3, \quad |\vec{c}| > 1.$$

cannot have a finite energy.

Hint: Using (c) and look at the energy of the solution in various balls.

2. Finite time extinction and hyper-contractiveity are important properties in modeling of some physical and biology systems. The essence of estimates is given by the following problem for ODE.

Assume $y(t) \ge 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \le \alpha - \beta y(t)^a$ for $\alpha > 0, \beta > 0$, then

(a) (10 points) For $a > 1$, $y(t)$ has the following hyper-contractive property

$$y(t) \le (\alpha/\beta)^{1/a} + \left[\frac{1}{\beta(a-1)t}\right]^{\frac{1}{a-1}}, \quad \text{for } t > 0.$$

(b) (2 points) For $a = 1$, $y(t)$ decays exponentially

$$y(t) \leq \alpha/\beta + y(0)e^{-\beta t}.$$

(c) (10 points) For $a < 1$, $\alpha = 0$, $y(t)$ has finite time extinction, which means that there exists T_{ext} such that $0 < T_{ext} \leq \frac{y^{1-a}(0)}{\beta(1-a)}$ and that $y(t) = 0$ for all $t > T_{ext}$.

3. Consider the speed v of a ball (density ρ, radius R) falling through a viscous fluid (density ρ_f, viscosity μ) with drag coefficient given by Stokes' law $\zeta = 6\pi R\mu$:

$$\frac{4}{3}\pi R^3 \rho \frac{dv}{dt} = \frac{4}{3}\pi R^3(\rho - \rho_f)g - \zeta v, \quad v(0) = v_0$$

(a): (5 points) Nondimensionalize the equation by writing, $v(t) = V\tilde{v}(\tilde{t})$ with $t = T\tilde{t}$. Select V, T (characteristic scales known as terminal velocity and settling time respectively) so that all coefficients in the ODE but one are equal to 1. Your equation will have a single dimensionless parameter given by the ratio of the initial speed v_0 to the characteristic speed V.
(b): (2 points) Solve the nondimensional problem for $\tilde{v}(\tilde{t})$.
(c): (8 points) Describe the behavior of the solution if the initial speed v_0 is (i) faster than and (ii) slower than the characteristic speed V. Compute the time to reach $(v_0 + V)/2$.

4. Let

$$V_h = \{v : v|_{I_j} \in P^k(I_j) \qquad 1 \leq j \leq N\}$$

where

$$I_j = (x_{j-1}, x_j), \qquad 1 \leq j \leq N$$

with

$$x_j = jh, \qquad h = \frac{1}{N}.$$

Here $P^k(I_j)$ denotes the set of polynomials of degree at most k in the interval I_j.

Recall the L^2 projection of a function $u(x)$ into the space V_h is defined by the unique function $u_h \in V_h$ which satisfies

$$||u - u_h|| \leq ||u - v|| \qquad \forall v \in V_h$$

where the norm is the usual L^2 norm. We assume $u(x)$ has at least $k + 2$ continuous derivatives.

(1) (5 points) Prove the error estimate

$$||u - u_h|| \leq Ch^{k+1}$$

Explain how the constant C depends on the derivatives of $u(x)$.

(2) (10 points) If another function $\varphi(x)$ also has at least $k+2$ continuous derivatives, prove

$$|\int_0^1 (u(x)-u_h(x))\varphi(x)dx| \leq Ch^{2k+2}$$

Explain how the constant C depends on the derivatives of $u(x)$ and $\varphi(x)$.

5. (15 points) Let $G(V,E)$ be a simple graph of order n and δ the minimum degree of vertices. Suppose that the degree sum of any pair of nonadjacent vertices is at least n and $F \subset E$ with $|F| \leq \lfloor \frac{\delta-2}{2} \rfloor$. Let $G-F$ be the graph obtained from G by deleting the edges in F. Prove that

(1) $G-F$ is connected and
(2) $G-F$ is Hamiltonian.

6. (15 points) Let $(F_n)_n$ be the Fibonacci sequence. Namely, $F_0 = 0, F_1 = 1, \ldots, F_{n+2} = F_{n+1} + F_n$.

Establish a relation between $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$ and F_n and use it to design an efficient algorithm that for a given n computes the n-th Fibonacci number F_n. In particular, it must be *more efficient* than computing F_n in n consecutive steps.

Give an estimate on the number of steps of your algorithm.

Hint: Not that if m is even then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^m = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m/2}\right)^2$$

and if m is odd then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^m = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and $m-1$ is even.