S.-T. Yau College Student Mathematics Contests 2019

Analysis and Differential Equations

Team (5 problems)

1) Show that there is no non-zero $f \in C_0^\infty(\mathbb{R}^2)$ (compactly supported smooth function) so that its Fourier transform $\widehat{f}(\xi)$ is also compactly supported.

2) Prove the following classical interior Schauder estimates:

   There exists a universal constant C, for all smooth compactly supported functions $u, f \in C_0^\infty(\mathbb{R}^3)$ with $\triangle u = f$, we have
   $$\|u\|_{C^{2,\alpha}} \leqslant C\|f\|_{C^{0,\alpha}},$$
   where $0 < \alpha < 1$ and $\|\cdot\|_{C^{k,\alpha}}$ are Hölder norms.

3) Let $(X, \mathcal{A}, \mu)$ be a probability space and let $T : X \to X$ be a measurable and measure preserving map, i.e., for all $A \in \mathcal{A}$, we have $\mu\big(T^{-1}(A)\big) = \mu(A)$. For $A, B \in \mathcal{A}$, if $\mu(A - B) = \mu(B - A) = 0$, we say that $A = B$ *a.e.*

   Assume $A \in \mathcal{A}$ such that $T^{-1}(A) = A$ *a.e.*.Prove that there exists a set $B \in \mathcal{A}$ so that $T^{-1}B = B$ and $A = B$ *a.e.*

4) Is there an entire function f with infinitely many zeroes, so that for every $r \in (0, 1)$, there exist constants $A_r, B_r < \infty$ such that
   $$|f(z)| \leqslant A_r e^{B_r|z|^r}$$
   for every $z \in \mathbb{C}$?

5) Let $u(t, x, y)$ be a smooth real function defined on $\mathbb{R} \times \mathbb{R}^2$ where $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$. We assume that it solves the following semilinear wave equation:
   $$-\frac{\partial^2}{\partial t^2}u + \triangle u = u^3.$$
   If the supports of the initial data $u(0, x)$ and $\dfrac{\partial u}{\partial t}(0, x)$ are compact, prove that, for all $t_0 \in \mathbb{R}$, the supports of $u(t_0, x)$ and $\dfrac{\partial u}{\partial t}(t_0, x)$ are compact.