S.-T. Yau College Student Mathematics Contests 2010

Analysis and Differential Equations

Individual

(Please select 5 problems to solve)

1. a) Let x_k, $k = 1, \ldots, n$ be real numbers from the interval $(0, \pi)$ and define $x = \dfrac{\sum_{i=1}^{n} x_i}{n}$. Show that

$$\prod_{k=1}^{n} \frac{\sin x_k}{x_k} \leq \left(\frac{\sin x}{x} \right)^n .$$

b) From

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

calculate the integral $\int_0^\infty \sin(x^2) dx$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that the set of points x in $\mathbb{R}$ where f is continuous is a countable intersection of open sets.

3. Let $f(z)$ be holomorphic in D: $|z| < 1$ and $|f(z)| \leq 1$ $(z \in D)$. If z_0 is a point in D such that both z_0 and $-z_0$ are zeros of order m of $f(z)$ and $0 < |z_0| \leq \dfrac{m-1}{m}$, then $|f(0)| < e^{-2}$.

4. Find a harmonic function f on the right half-plane such that when approaching any point in the positive half of the y-axis, the function has limit 1, while when approaching any point in the negative half of the y-axis, the function has limit -1.

5. Let $K(x, y) \in L^1([0, 1] \times [0, 1])$. For all $f \in C^0[0, 1]$, the space of continuous functions on $[0, 1]$, define a function

$$Tf(x) = \int_0^1 K(x, y) f(y) dy$$

Prove that $Tf \in C^0([0, 1])$. Moreover $\Omega = \{Tf| \; ||f||_{sup} \leq 1\}$ is pre-compact in $C^0([0, 1])$, i.e. every sequence in Ω has a converging subsequence, here $||f||_{sup} = \sup\{|f(x)||x \in [0, 1]\}$. (Hint: Every Lebesgue integrable function over the square can be approximated by polynomial functions in the L^1 norm.)

6. Consider the equation $\dot{x} = -x + f(t, x)$, where $|f(t, x)| \leq \phi(t)|x|$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\int^{\infty} \phi(t)dt < \infty$. Prove that every solution approaches zero as $t \to \infty$.