**INDIVIDUAL TEST**
**S.-T YAU COLLEGE MATH CONTESTS 2012**

# Analysis and Differential Equations

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

**1.** Compute the integral

$$\int_0^\infty \frac{x^p}{1+x^2}dx, -1 < p < 1.$$

**2.** Construct a one to one conformal mapping from the region

$$U = \{z \in \mathbb{C}||z - \frac{i}{2}| < \frac{1}{2}\}/\{z||z - \frac{i}{4}| < \frac{1}{4}\}$$

onto the unit disk.

**3.** Let $f(x)$ be a nonlinear $C^2$ function on $\mathbb{R}$. Show that

$$\sup |f'(x)|^2 \leq 4 \sup |f(x)| \sup |f''(x)|.$$

**4.** Let $f(x)$ be a real measurable function defined on $[a, b]$. Let $n(y)$ be the number of solutions of the equation $f(x) = y$. Prove that $n(y)$ is a measurable function on $\mathbb{R}$.

**5.** For $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, let $g$ in $L^q$. Consider the linear functional $F$ on $L^p$ given by: $F(f)$ is equal to the integral of $fg$. Show that $||F|| = ||g||_q$.

**6.** Let $\mathbb{R}^n_+ = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n | x_n > 0\}$. Show that the formula

$$u(x) = \frac{2x_n}{n\alpha_n} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy, x \in \mathbb{R}^n_+$$

is a solution of the problem

$$\Delta u = 0, \text{ in } \mathbb{R}^n_+, u = g \text{ on } \partial\mathbb{R}^n_+,$$

where $\alpha_n$ is the volume of the unit n dimensional sphere.