S.-T. Yau College Student Mathematics Contests 2022

# Analysis and Differential Equations

*Solve every problem.*

**Problem 1.** For $n \geq 1$, we consider the integral

$$I_n = \int_{[0,1]^n} \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \, dx_1 \cdots dx_n.$$

Prove that $\lim\limits_{n\to\infty} I_n$ exists.

**Problem 2.** Let $U \subset \mathbf{C}$ be a non-empty open set and $f : U \to U$ be a non-constant holomorphic function. Prove that, if $f \circ f = f$, then $f(z) \equiv z$ for all $z \in U$.

**Problem 3.** Let $X \subset \mathbf{R}$ be a set with positive (Lebesgue) measure. Show that we can find an arithmetic progression of 2022 terms in $X$, *i.e.*, there exists $x_1, \ldots, x_{2022} \in X$ so that the $x_{i+1} - x_i$'s are all equal and positive, $i = 1, \ldots, 2021$.

**Problem 4.** Let $C([0,1])$ be the space of all continuous $\mathbf{C}$-valued functions equipped with $L^\infty$-norm. Let $\mathbf{P} \subset C([0,1])$ be a closed linear subspace. Assume that the elements of $\mathbf{P}$ are polynomials. Prove that $\dim \mathbf{P} < \infty$.

**Problem 5.** Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary. Assume that $u \in C\left(\overline{\mathbf{R}^3 - \Omega}\right)$ is a harmonic function on $\mathbf{R}^3 - \Omega$ so that $u\big|_\Omega = 1$ and $\lim\limits_{|x|\to\infty} |u(x)| = 0$.

Prove that for such $u$, $\lim\limits_{|x|\to\infty} |x|u(x)$ exists.

**Problem 6.** Let $f(x,y) \in C^1(\mathbf{R}^2)$. We assume that there exists $C > 0$ so that for all $(x,y) \in \mathbf{R}^2$, $\left|\frac{\partial f}{\partial y}(x,y)\right| \leq C$. Prove that the following ODE has a globally defined solution for all $y(0) = y_0 \in \mathbf{R}$:

$$\begin{cases} \frac{d}{dx} y(x) = f(x, y(x)), \\ y(0) = y_0. \end{cases} \tag{1}$$

In addition, we assume that $f$ is 1-periodic in $x$, *i.e.*, for all $(x,y) \in \mathbf{R}^2$, we have $f(x+1,y) = f(x,y)$. Prove that if (1) admits a globally defined bounded solution, then (1) admits a periodic solution.