S.-T. Yau College Student Mathematics Contests 2020

Computational and Applied Mathematics

Solve every problem.

Problem 1. Let $f \in C^{k+1}[-1,1]$ and $[-1,1]$ be partitioned into subintervals $I_j = [(j-1)h, jh]$ of width h. Assume p is a polynomial of degree k which approximates f in I_j with

$$\max_{x \in I_j} \left| p_j(x) - f(x) \right| \le C_0 h^{k+1},$$

where C_0 is a constant independent of j. Show that there exists an another constant C, independent of j, such that

$$\max_{x \in I_{j\pm 1}} \left| p_j(x) - f(x) \right| \le C h^{k+1}.$$

(as long as $I_{j\pm 1} \subset [-1,1]$, of course).

Problem 2. Consider the iteration

$$x_{n+1} = x_n - \left(\frac{x_n - x_0}{f(x_n) - f(x_0)} \right) f(x_n)$$

for finding the roots of a two times continuous differentiable function $f(x)$. Assuming the method converges to a simple root x^*, what is the rate of convergence? Justify your answer.

Problem 3. Suppose $\mathbf{A}$ is an $m \times m$ matrix with a complete set of orthonormal eigenvectors $\mathbf{q_1}, \dots, \mathbf{q_m}$ and corresponding eigenvalues $\lambda_1, \dots, \lambda_m$. Assume that $|\lambda_1| > |\lambda_2| > |\lambda_3|$ and $\lambda_j \ge \lambda_{j+1}$ for $j = 3, \dots, m$. Consider the power method $\mathbf{v}^{(k)} = \mathbf{A}\mathbf{v}^{(k-1)}/\lambda_1$, with $\mathbf{v}^{(0)} = \alpha_1 \mathbf{q}_1 + \cdots + \alpha_m \mathbf{q}_m$ where α_1 and α_2 are both nonzero. Show that the sequence $\{\mathbf{v}^{(k)}\}_{k=0}^{\infty}$ converges linearly to $\alpha_1 \mathbf{q}_1$ with asymptotic constant $C = |\lambda_2/\lambda_1|$.

Problem 4. For the initial value problem $y' = f(t,y), \quad y(0) = y_0$ on the interval $[0,\ T]$, consider the implicit two-step method

$$\begin{aligned} y_{n+1} &= \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2h}{3} f(t_{n+1}, y_{n+1}), \\ y_1 &= y_0 + h f(t_1, y_0), \end{aligned}$$

where h is the step size and $t_n = nh$.

(a) What is the order of the accuracy of the scheme?

(b) Check the stability of the scheme by analyzing the stability polynomial?

(c) Find the stability region of the scheme.

Problem 5. Suppose the difference scheme $u^{n+1} = Bu^n$ is stable, and $C(\Delta t)$ is a bounded family of operators. Show that the scheme

$$u^{n+1} = (B + \Delta t C(\Delta t))u^n$$

is stable.

Problem 6. Let A be an $m \times m$ nonsingular matrix. Suppose $\inf_{p_n \in P^n} ||p_n(A)|| = ||p^*(A)|| > 0$ where P^n denotes the set of all degree-n monic polynomials:

$$P^n = \{p : p \text{ is a polynomial of degree } n, \, p(z) = z^n + \cdots\}.$$

Prove that p^* is unique.