S.-T. Yau College Student Mathematics Contests 2015

Applied Math. and Computational Math.

Team (5 problems)

Problem 1. Consider the elliptic interface problem

$$(a(x)u_x)_x = f,\ x \in (0,1)$$

with the Dirichlet boundary condition

$$u(0) = u(1) = 0.$$

Here, f is a smooth function, the elliptic coefficient $a(x)$ is discontinuous across an interface point ξ, that is,

$$a(x) = \begin{cases} a_0 & \text{for } 0 < x < \xi \\ a_1 & \text{for } \xi < x < 1, \end{cases}$$

$a_0, a_1 > 0$ are positive constants, and $0 < \xi < 1$ is an interface point. Across the interface, we need to impose two jump conditions

$$u(\xi-) = u(\xi+),\ a(\xi-)u_x(\xi-) = a(\xi+)u_x(\xi+).$$

Question:

1. (25%) Design a numerical method to solve this problem. The method should be at least first order. It is better to be high order (if your method is first order, you get 20% points).
2. (75%) Prove your accuracy and convergence arguments (if your method is first order, you get 60% points).

Problem 2. Let G be graph of a social network, where for each pair of members there is either no connection, or a positive or a negative one.

An unbalanced cycle in G is a a cycle which have odd number of negative edges. Traversing along such a cycle with social rules such as friend of enemy are enemy would result in having a negative relation of one with himself!

A resigning in G at a vertex v of G is to switch the type (positive or negative) of all edges incident to v.

Question: Show that one can switch all edge of G into positive edges using a sequence resigning if and only if there is no unbalanced cycle in G.

Problem 3. We consider particles which are able to produce new particles of like kind. A single particle forms the original, or zero, generation. Every particle has probability p_k ($k = 0, 1, 2, \dots$) of creating exactly k new particles; the direct descendants of the nth generation form the $(n + 1)$st generation. The particles of each generation act independently of each other.

Assume $0 < p_0 < 1$. Let $P(x) = \sum_{k\geq 0} p_k x^k$ and $\mu = P'(1) = \sum_{k\geq 0} kp_k$ be the expected number of direct descendants of one particle. Prove that if $\mu > 1$, then the probability x_n that the process terminates at or before the nth generation tends to the unique root $\sigma \in (0,1)$ of equation $\sigma = P(\sigma)$.

Problem 4. (Isopermetric inequality). Consider a closed plane curve described by a parametric equation $(x(t), y(t)), 0 \leq t \leq T$ with parameter t oriented counterclockwise and $(x(0), y(0)) = (x(T), y(T))$.

(a): Show that the total length of the curve is given by

$$L = \int_0^T \sqrt{(x'(t))^2 + (y'(t))^2)}\, dt$$

(b): Show that the total area enclosed by the curve is given by

$$A = \frac{1}{2} \int_0^T \big(x(t)y'(t) - y(t)x'(t)\big)\, dt$$

(c): The classical iso-perimetric inequality states that for closed plane curves with a fixed length L, circles have the largest enclosed area A. Formulate this question into a variational problem.

(d): Derive the Euler-Lagrange equation for the variational problem in **(c)**.

(e): Show that there are two constants x_0 and y_0 such that

$$(x(t) - x_0)^2 + (y(t) - y_0)^2 \equiv r^2$$

where $r = L/(2\pi)$. Explain your result.

Problem 5. Let $A \in \mathbb{R}^{n\times m}$ with rank $r < \min(m,n)$. Let $A = U\Sigma V^T$ be the SVD of A, with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

(a) Show that, for every $\epsilon > 0$, there is a full rank matrix $A_\epsilon \in \mathbb{R}^{n\times m}$ such that $||A - A_\epsilon||_2 = \epsilon$.

(b) Let $A_k = U\Sigma_k V^T$ where $\Sigma_k = \mathrm{diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0)$ and $1 \leq k \leq r-1$. Show that $\mathrm{rank}(A_k) = k$ and

$$\sigma_{k+1} = ||A - A_k||_2 = \min\{||A - B||_2 \quad | \quad \mathrm{rank}(B) \leq k\}$$

(c) Assume that $r = \min(m,n)$. Let $B \in \mathbb{R}^{n\times m}$ and assume that $||A - B||_2 < \sigma_r$. Show that $\mathrm{rank}(B) = r$.