S.-T. Yau College Student Mathematics Contests 2010

Analysis and Differential Equations

Team

(Please select 5 problems to solve)

1. a) Let $f(z)$ be holomorphic in D: $|z| < 1$ and $|f(z)| \leq 1$ ($z \in D$). Prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}. \qquad (z \in D)$$

b) For any finite complex value a, prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |a - e^{i\theta}| d\theta = \max\{\log |a|, 0\}.$$

2. Let $f \in C^1(\mathbf{R}), f(x+1) = f(x)$, for all x, then we have

$$||f||_\infty \leq \int_0^1 |f(t)|dt + \int_0^1 |f^{'}(t)|dt.$$

3. Consider the equation

$$\ddot{x} + (1 + f(t))x = 0.$$

We assume that $\int^\infty |f(t)|dt < \infty$. Study the Lyapunov stability of the solution $(x, \dot{x}) = (0, 0)$.

4. Suppose $f : [a, b] \to \mathbf{R}$ be a L^1-integrable function. Extend f to be 0 outside the interval $[a, b]$. Let

$$\phi(x) = \frac{1}{2h} \int_{x-h}^{x+h} f$$

Show that

$$\int_a^b |\phi| \leq \int_a^b |f|.$$

5. Suppose $f \in L^1[0, 2\pi], \hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx$, prove that

1) $\displaystyle\sum_{|n|=0}^{\infty} |\hat{f}(n)|^2 < \infty$ implies $f \in L^2[0, 2\pi]$,

2) $\displaystyle\sum_n |n\hat{f}(n)| < \infty$ implies that $f = f_0, a.e., f_0 \in C^1[0, 2\pi]$,

where $C^1[0, 2\pi]$ is the space of functions f over $[0, 1]$ such that both f and its derivative $f^{'}$ are continuous functions.

6. Let Ω be a bounded domain of $\mathbf{R}^n$ and let f be a smooth function defined in $[0, +\infty)$ such that $f(t)/t$ is strictly decreasing. Assume that u_1 and u_2 are positive solutions of

$$\Delta u + f(u) = 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega$$

Show that $u_1 = u_2$. (Hint: Calculate $\Delta \log \frac{u_2}{u_1}$ and consider the maximum principle.)