S.-T. Yau College Student Mathematics Contests 2019

Algebra and Number Theory

Team (5 problems)

1) Let S_n be the group of permutations of $\{1, 2, ..., n\}$. Let $\sigma \in S_n$ be the permutation

$$(1, n)(2, n-1) \cdots (k, n-k+1) \cdots .$$

Prove that the centraliser $Z_{S_n}(\sigma)$ is isomorphic to $S_{[\frac{n}{2}]} \ltimes (\mathbb{Z}/2\mathbb{Z})^{[\frac{n}{2}]}$.

2) Recall that the algebra of regular functions on a vector space W is the symmetric algebra of linear forms on W.

Let $V = \mathbb{C}^2$. Let $\mathbb{C}[\mathrm{End}_{\mathbb{C}}(V)]$ be the algebra of regular functions on $\mathrm{End}_{\mathbb{C}}(V)$. The natural action of the group $G = SL_2(\mathbb{C})$ on V induces an action of $G \times G$ on $\mathrm{End}_{\mathbb{C}}(V)$ by left and right multiplication. Thus we get an action of $G \times G$ on $\mathbb{C}[\mathrm{End}_{\mathbb{C}}(V)]$.

Compute the algebra of fixed points $\mathbb{C}[\mathrm{End}_{\mathbb{C}}(V)]^{G \times G}$.

3) Let R be a Noetherian ring and $I \subset R$ be an ideal. Define the Rees algebra as

$$Rees(I, R) := \bigoplus_{n \geqslant 1} I^n t^n \subset R[t].$$

Prove that $Rees(I, R)$ is Noetherian.

4) Let p be a odd prime. Let Φ_p be the p-th cyclotomic field, *i.e.*, $\Phi_p = \mathbb{Q}(\zeta_p)$ where ζ_p is a primitive p-th root of unity.

   1. Show that $\Phi_p/\mathbb{Q}$ is a Galois extension with Galois group $(\mathbb{Z}/p\mathbb{Z})^\times$.
   2. Deduce that Φ_p contains a unique quadratic extension of $\mathbb{Q}$.
   3. Write g_p for the Gauss sum $\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^a$. Show that
      - $\overline{g_p} = \left(\frac{-1}{p}\right) g_p$,
      - $|g_p|^2 = p$.
   4. Determine the unique quadratic extension of $\mathbb{Q}$ contained in Φ_p.

5)
   1. Let E/F be a finite Galois extension. Assume that the Galois group $Gal(E/F)$ is generated by a single element σ. Let x be an element of E such that $tr_{E/F}(x) = 0$. Show that there exists $y \in E$ such that $x = \sigma(y) - y$.
   2. Let F be a field of characteristic p, and let E/F be a Galois extension of degree p. Show that there exists $x \in F$ such that $E \cong F[T]/(T^p - T - x)$.