S.-T. Yau College Student Mathematics Contests 2014

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Calculate the integral:

$$\int_0^\infty \frac{\log x}{1+x^2} dx.$$

2. Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is precisely $\mathbb{Q}$.

3. Prove that any bounded analytic function F over $\{z|r < |z| < R\}$ can be written as $F(z) = z^\alpha f(z)$, where f is an analytic function over the disk $\{z||z| < R\}$ and α is a constant.

4. Let $D \subset \mathbb{R}^n$ be a bounded open set, $f : \bar{D} \to \bar{D}$ is a smooth map such that its Jacobian $\left|\frac{\partial f}{\partial x}\right| \equiv 1$, where $\bar{D}$ denotes the closure of D. Prove

(a) for each small ball $B_\epsilon(x)$, there exists a positive integer k such that $f^k(B_\epsilon(x)) \cap B_\epsilon(x) \neq \varnothing$, where $B_\epsilon(x)$ denotes the ball centered at x with radius ϵ;

(b) there exists $x \in \bar{D}$ and a sequence $k_1, k_2, \cdots k_j, \cdots$ such that $f^{k_j}(x) \to x$ as $k_j \to \infty$.

5. Let u be a subharmonic function over a domain $\Omega \subset \mathbf{C}$, i.e., it is twice differentiable and $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0$. Prove that u achieves its maximum in the interior of Ω only when u is a constant.

6. Suppose that $\phi \in C_0^\infty(\mathbf{R}^n)$, $\int_{\mathbf{R}^n} \phi dx = 1$. Let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon), x \in \mathbf{R}^n, \epsilon > 0$. Prove that if $f \in L^p(\mathbf{R}^n), 1 \leq p < \infty$, then $f * \phi_\epsilon \to f$ in $L^p(\mathbf{R}^n)$, as $\epsilon \to 0$. It is not true for $p = \infty$.