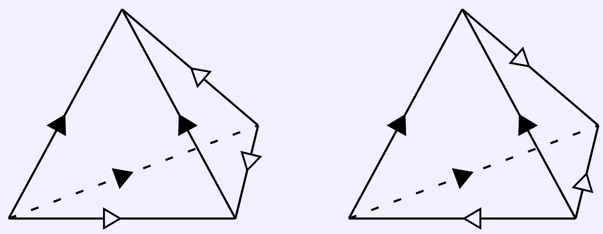
S.-T. Yau College Student Mathematics Contests 2022

Geometry and Topology

Solve every problem.

Problem 1. The topological space X is obtained by gluing two tetrahedra as illustrated by the figure. There is a unique way to glue the faces of one tetrahedron to the other so that the arrows are matched. The resulting complex has 2 tetrahedra, 4 triangles, 2 edges and 1 vertex.

Show that X can not have the homotopy type of a compact manifold without boundary.



Problem 2. Suppose (M, h) is a closed (*i.e.*, compact without boundary) Riemannian manifold, and h is a metric on M with $\sec(h) \leq -1$, where $\sec(h)$ is the sectional curvature. Suppose Σ is a closed minimal surface with genus g in (M, h). Show that

$$\mathrm{Area}(\Sigma) \leq 4\pi(g-1).$$

Remark: *A minimal surface is an immersed surface with constant mean curvature* 0.

Problem 3. For any topological space X, the *n-th symmetric product* of X is the quotient of the Cartesian product $(X)^n$ by the action of the symmetric group S_n, which permutes the factors in $(X)^n$. This space is denoted by $\mathrm{SP}^n(X)$, and the topology is the natural quotient topology induced from $(X)^n$.

Show that $\mathrm{SP}^n(\mathbf{CP}^1)$ is homeomorphic to $\mathbf{CP}^n$. Here $\mathbf{CP}^1$ and $\mathbf{CP}^n$ are equipped with the manifold topology.

Problem 4. Let M be a complete noncompact Riemannian manifold. M is said to have the *geodesic loops to infinity property* if for any $[\alpha] \in \pi_1(M)$ and any compact subset $K \subset M$, there is a geodesic loop $\beta \subset M \backslash K$, such that β is homotopic to α.

Show that if a complete noncompact Riemannian manifold M does not have the geodesic loops to infinity property, then there is a line in the universal cover $\widetilde{M}$.

Remark: *A line is a geodesic* $\gamma : (-\infty, \infty) \to M$ *such that* $\mathrm{dist}(\gamma(s), \gamma(t)) = |s-t|$; *a geodesic loop is a curve* $\beta : [0, 1] \to M$ *that is a geodesic and* $\beta(0) = \beta(1)$.

Problem 5. A topological space X is called an *H-space* if there exist $e \in X$ and $\mu : X \times X \to X$ such that $\mu(e, e) = e$ and the maps $x \to \mu(e, x)$ and $x \to \mu(x, e)$ are both homotopic to the identity map.

(a) Show that the fundamental group of an H-space is Abelian.

(b) Show that the sphere S^{2022} is not an H-space.

Historic Remark: *"H" was suggested by Jean-Pierre Serre in recognition of the contributions in Topology by Heinz Hopf.*

Problem 6. A hypersurface $\Sigma \subset \mathbf{R^{n+1}}$ is called a *shrinker* if it satisfies the equation

$$H(x) = \frac{1}{2}\langle x, \vec{n}\rangle.$$

Here H is the mean curvature, which is $-\langle \mathrm{tr}_A, \vec{n}\rangle$ where A is the second fundamental form, x is the position vector, and $\vec{n}$ is outer unit normal vector.

(a) Show that $S^n(\sqrt{2n})$, the sphere with radius $\sqrt{2n}$, is a shrinker.

(b) Show that any compact shrinker without boundary must intersect with $S^n(\sqrt{2n})$.