S.-T. Yau College Student Mathematics Contests 2023

Computational and Applied Mathematics

Solve every problem.

1. Consider the forward and the centered finite difference formulas

$$D_h^+ f(x_0) = \frac{f(x_0+h) - f(x_0)}{h}, \tag{1}$$

$$D_h^0 f(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}, \tag{2}$$

to approximate the derivative of f at a point x_0. Assume f is a smooth function in a neighborhood of x_0 containing the points $x_0 + h$ and $x_0 - h$.

(a) Prove that $D_h^+ f(x_0)$ and $D_h^0 f(x_0)$ approximate $f'(x_0)$ to $O(h)$ and $O(h^2)$, respectively.

(b) Derive an $O(h^2)$ approximation to $f'(x_0)$ from $D_h^+ f(x_0)$ by doing Richardson extrapolation.

(c) Take $f(x) = \sin x$ and $x_0 = 0$. Prove that both $D_h^+ f(x_0)$ and $D_h^0 f(x_0)$ converge quadratically to $f'(x_0)$ as $h \to 0$ and that in fact they produce the same approximation to $f'(x_0)$ in this particular case.

2. For functions defined on a closed interval $[0, 1]$, we want to compute the following definite integral,

$$I[f] = \int_0^1 f(x) \log(1/x) dx.$$

Here we consider the weight function $\log(1/x)$, and denote $P_n(x)$ as the monic orthogonal polynomials for the corresponding weighted inner product.

(a) Let $P_0 = 1$. Find $P_1(x)$, and the corresponding node x_1^1 and weight ω_1^1 for the 1-point Gaussian quadrature rule.

(b) Derive a recursive formula for $P_{n+1}(x)$ using $P_n(x)$ and $P_{n-1}(x)$.

(c) Consider the normalized orthogonal polynomials $Q_n(x) = P_n(x)/||P_n||$, where

$$||P_n|| = \sqrt{P_n(x)^2 \log(1/x) dx}.$$

Derive a recursive formula for $Q_{n+1}(x)$ using $Q_n(x)$ and $Q_{n-1}(x)$.

(d) Use the above recursive formula to show that $x = \lambda$ is a node of the 4-point Gaussian quadrature if and only if it is an eigenvalue of a symmetric, tridiagonal matrix. Write out the form of the symmetric and tridiagonal matrix explicitly.

3. Let A be a real $n \times n$ matrix with distinct eigenvalues such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \geq 0,$$

with corresponding eigenvectors $\{v_j\}_{j=1}^n$.

(a) Show that the power iteration

$$z_m = \frac{A^m z_0}{||A^m z_0||_\infty} \longrightarrow \pm \frac{v_1}{||v_1||_\infty}, \quad \forall z_0 \in \mathbb{R}^n.$$

(b) Consider the following iteration with initial guess $x_0 = y_0 = 1$,

$$x_{n+1} = x_n + y_n, \quad y_{n+1} = x_{n+1} + x_n.$$

Show that $y_n/x_n \to \sqrt{2}$ as $n \to \infty$.

4. Consider the initial value problem

$$y' = f(t, y), \quad 0 < t \leq T. \tag{3}$$
$$y(0) = y_0. \tag{4}$$

Assume f is continuous and Lipschitz in y in $[0, T] \times (-\infty, \infty)$. Denote $y_n \approx y(t_n)$, $t_n = nh$, and $h = T/N$, with N a positive integer, and consider the one-step method

$$y_{n+1} = y_n + \alpha h f(t_n, y_n) + \beta h f(t_n + \gamma h, y_n + \gamma h f(t_n, y_n)),$$

where α, β and γ are real parameters.

(a) Prove that the method is consistent if and only if $\alpha + \beta = 1$, and the order of the method can not exceed 2.

(b) Suppose that a second-order method of the above form is applied to $f(t, y) = -\lambda y$ with $\lambda > 0$, and the initial condition $y_0 = 1$. Show that the sequence $(y_n)_{n \geq 0}$ is bounded if and only if $h \leq \frac{2}{\lambda}$. Show further that for such h,

$$|y(t_n) - y_n| \leq \frac{1}{6} \lambda^3 h^2 t_n, \quad n \geq 0.$$

5. Let $u(t,x)$ be the solution of the initial-boundary value problem

$$u_t = Du_{xx}, \quad 0 < x < L, \quad 0 < t \le T, \tag{5}$$
$$u(0,x) = f(x) \tag{6}$$
$$u(t,0) = u(t,L) = 0, \tag{7}$$

where $L > 0$ and $D > 0$. Consider the finite difference scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}, \quad j = 1, \ldots, M-1, \quad n = 0, 1, \ldots, N-1 \tag{8}$$

with $u_0^n = u_M^n = 0$ for all n and $u_j^0 = f(j\Delta x)$, $j = 0, \ldots, M$. Here $\Delta t = T/N$ and $\Delta x = L/M$ and $u_j^n \approx u(n\Delta t, j\Delta x)$.

(a) Prove that (8) is consistent with (5).

(b) Prove that if $\Delta t \le \frac{1}{2D}(\Delta x)^2$ the finite difference scheme (8) is stable under the l^∞ norm.

(c) Prove that if $\Delta t \le \frac{1}{2D}(\Delta x)^2$ the finite difference scheme (8) converges in the l^∞ norm to the exact solution of (5)-(7).

6. Let $\psi^\varepsilon(t,x)$ be the solution to the following Schrödinger equation:

$$\mathrm{i}\varepsilon\frac{\partial \psi^\varepsilon}{\partial t} = -\frac{\varepsilon^2}{2}\nabla_x^2\psi^\varepsilon + V(x)\psi^\varepsilon, \quad x = (x_1, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n,$$

where $\mathrm{i} = \sqrt{-1}$, $\varepsilon \ll 1$ is a small positive real number (rescaled Planck constant), $\nabla_x^2 = \sum_{j=1}^n \partial_{x_j}^2$, and $V(x) \in C^\infty(\mathbb{R}^n)$ is the potential function.

Consider the WKB expansion

$$\psi^\varepsilon(t,x) = A(t,x)e^{\mathrm{i}\frac{S(t,x)}{\varepsilon}},$$

(a) Derive equations for $A(t,x)$ and $S(t,x)$ by asymptotic expansion. (Here both $A(t,x)$ and $S(t,x)$ are real-valued functions, and do not depend on ε.)

(b) Define $u(t,x) = \nabla_x S(t,x) \in \mathbb{R}^n$. Derive an equation for $u(t,x)$. Suppose $u(0,x) \in C^\infty(\mathbb{R}^n)$, will $u(t,x)$ always be in $C^\infty(\mathbb{R}^n)$ for all $t > 0$? Explain why.