Syllabus on Geometry and Topology

This syllabus is organized into two main parts: *Differential Geometry* and *Algebraic Topology*. The references at the end are also regrouped by subject to make them easier for students to use.

Part I. Differential Geometry

1. Smooth manifolds and differential topology

- Inverse function theorem
- Implicit function theorem
- Submanifolds
- Sard's theorem
- Embedding theorem
- Transversality
- Degree theory
- Integration on manifolds

2. Matrix Lie groups over $\mathbb{R}$ and $\mathbb{C}$

- Definitions of $GL(n)$, $SU(n)$, $SO(n)$, and $U(n)$
- Their manifold structures
- Lie algebras
- Right- and left-invariant vector fields
- Right- and left-invariant differential forms
- The exponential map

3. Vector bundles

- Real and complex vector bundles
- Tangent and cotangent bundles
- Dual bundles
- Tensor products
- Exterior products
- Direct sums
- Pull-back bundles

4. Differential forms and cohomology

- Differential forms
- Exterior product
- Exterior derivative

- de Rham cohomology
- Behavior under pull-back

5. **Metrics and Riemannian geometry**
   - Metrics on vector bundles
   - Riemannian metrics
   - Definition of a geodesic
   - Existence and uniqueness of geodesics
6. **Principal bundles and associated bundles**
   - Principal Lie group bundles for matrix groups
   - Associated vector bundles
   - Relation between principal bundles and vector bundles
7. **Connections and curvature**
   - Covariant derivative on a vector bundle
   - Connection on a principal bundle
   - Relation between the two notions
   - Curvature
   - Flat connections
   - Parallel transport
8. **Curvature and global Riemannian geometry**
   - Levi–Civita connection
   - Properties of the Riemann curvature tensor
   - Manifolds of constant curvature
   - Jacobi fields
   - Second variation of geodesics
   - Manifolds of nonpositive curvature
   - Manifolds of positive curvature

Part II. Algebraic Topology

1. **Fundamental group and covering spaces**
   - Fundamental groups
   - Covering spaces
2. **Homotopy theory**
   - Higher homotopy groups

- Fibrations
- Long exact sequence of a fibration

3. **Homology and cohomology**
   - Singular homology and cohomology
   - Relative homology
   - Čech cohomology
   - de Rham cohomology
   - Equivalence between singular, Čech, and de Rham cohomology
4. **CW complexes and computational tools**
   - CW complexes
   - Homology of CW complexes
   - Mayer–Vietoris sequence
   - Universal coefficient theorem
   - Künneth formula
5. **Duality and fixed point theory**
   - Poincaré duality
   - Lefschetz fixed point formula
   - Hopf index theorem

References

A. Differential Topology and Smooth Manifolds

- V. Guillemin and A. Pollack, *Differential Topology.*
- J. Milnor, *Topology from the Differentiable Viewpoint.*

B. Differential Geometry and Riemannian Geometry

- C. Taubes, *Differential Geometry: Bundles, Connections, Metrics and Curvature.*
- J. Lee, *Introduction to Riemannian Manifolds*, 2nd ed.
- S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry.*
- M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, updated edition.
- W. Kühnel, *Differential Geometry: Curves, Surfaces, Manifolds*, 2nd ed.
- A. Gray, *Tubes*, 2nd ed.
- P. Petersen, *Riemannian Geometry*, 4th ed.

- B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry — Methods and Applications, Part I: The Geometry of Surfaces, Transformation Groups, and Fields*.
- 陈维桓, 微分几何引论 (2013), 高等教育出版社.

C. Differential Forms and Bundles

- M. P. do Carmo, *Differential Forms and Applications*.
- A. Hatcher, *Vector Bundles and K-Theory* (online available).

D. Algebraic Topology

- A. Hatcher, *Algebraic Topology*.
- W. Fulton, *Algebraic Topology*.
- E. Spanier, *Algebraic Topology*.
- M. Greenberg and J. Harper, *Algebraic Topology: A First Course*.
- A. Hatcher, *Algebraic Topology* (online available), Cambridge University Press.
- B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry — Methods and Applications, Part II: The Geometry and Topology of Manifolds*.
- B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry — Methods and Applications, Part III: Introduction to Homology Theory*.

Remark for students. The references are not meant to be read cover to cover. A good strategy is to use one primary textbook for each subject and consult the others for examples, alternative proofs, and additional topics.